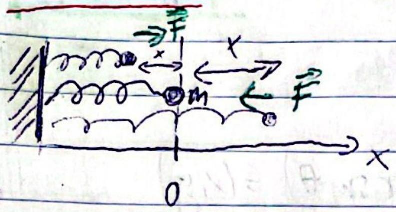
# Advanced Mechanics

## Harmonic Oscillator

### Free 1D – no damping/driving




~~resting~~ equilibrium position

restoring force — stiffness — const.

Hooke's law: $\boxed{\hat{x}\cdot\vec{F}(x) = -Kx = F_x}$

↳ approx. working for 'small' displacement

$$-kx = F(x) = ma = m\frac{d^2x}{dt^2} = m\ddot{x} \Rightarrow$$

ODE — Equation of motion: $\boxed{\ddot{x}(t) + \frac{K}{m}x(t) = 0}$, $\frac{K}{m} = \omega_0^2$ = natural freq.

① if $x(t)$ is a sol. then $C\cdot x(t)$ is also a sol.

② if $x_1, x_2$ are sols, also their LC is a sol.

③ if $x_1, x_2$ lin. indep. $\Rightarrow x_{gen}(t) = A_1x_1(t) + A_2x_2(t)$ — from initial conditions

**Solutions to** $\ddot{x} + \omega_0^2 x = 0$:

↳ 3 ways of expressing

$x_{1,2} = e^{\pm i\omega_0 t} \Rightarrow$ ① $\boxed{x(t) = A_1e^{i\omega_0 t} + A_2e^{-i\omega_0 t}}$ $\in \mathbb{R}$, $A_{1,2} \in \mathbb{C}$

Euler formula: $\boxed{e^{\pm iy} = \cos(y) \pm i\sin(y)}$ to switch between forms:

② $\boxed{x(t) = B_1\cos(\omega_0 t) + B_2\sin(\omega_0 t)}$ $B_{1,2} \in \mathbb{C}$

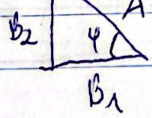
↳ $\boxed{\begin{aligned} A_1 &= B_1 + B_2 \\ A_2 &= B_1 - iB_2 \end{aligned}}$

A, B: Constants come from initial conditions: $x(0) = 0 = B_1$
$\Rightarrow x(t) = B_2\sin(\omega_0 t)$

③ Or $\boxed{x(t) = A\cos(\omega_0 t - \varphi)}$ with constants $\underline{\underline{A}}$ and $\underline{\underline{\varphi}}$

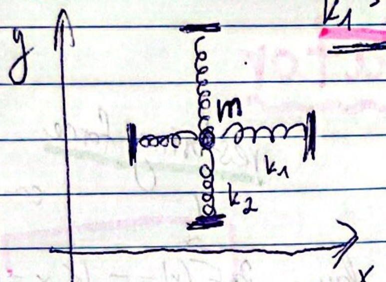
can be obtained from form ② as:

$$x(t) = A\left(\frac{B_1}{A}\cos(\omega_0 t) + \frac{B_2}{A}\sin(\omega_0 t)\right) = A(\cos(\varphi)\cos(\omega_0 t) + \sin(\varphi)\sin(\omega_0 t))$$
$$= A\cdot\cos(\omega_0 t - \varphi)$$




$A\cos\varphi = B_1$
$A\sin\varphi = B_2$

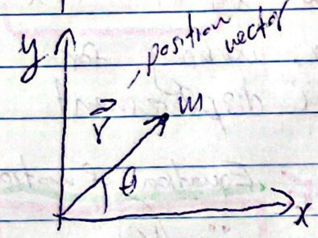
## Free 2D




$k_1 = k_2 = k$ – both springs have same spring const.

$= $ isotropic

$$\Rightarrow \quad \boxed{\begin{aligned} F_x &= -kx \\ F_y &= -ky \end{aligned}} \quad \begin{aligned} &= m\ddot{x} \\ &= m\ddot{y} \end{aligned}$$




$$\vec{r} = (\underbrace{r}_{|\vec{r}|}\cos\theta,\ r\sin\theta) = (x, y)$$

decompose

Equations of motion: $$\begin{aligned} \ddot{x} + \omega_0^2 x &= 0 \\ \ddot{y} + \omega_0^2 y &= 0 \end{aligned}$$

$\Rightarrow$ solution:
$$\boxed{\begin{aligned} x &= A\cos(\omega_0 t - \alpha) \\ y &= B\cos(\omega_0 t - \beta) \end{aligned}} \quad \rightsquigarrow \cos(\omega_0 t - \alpha) = \frac{x}{A}$$

$$y = B\cos(\omega_0 t - \beta - \alpha + \alpha) = B\cos[(\omega_0 t - \alpha) + (\alpha - \beta)] =$$
$$= B[\cos(\omega_0 t - \alpha)\cos(\alpha - \beta) - \sin(\omega_0 t - \alpha)\sin(\alpha - \beta)] =$$
$$= \frac{B}{A}[x\cos(\underbrace{\alpha - \beta}_{\delta}) - \underbrace{A\sin(\omega_0 t - \alpha)}\sin(\underbrace{\alpha - \beta}_{\delta})] =$$

$$x^2 = A^2\cos^2(\omega_0 t - \alpha)$$
$$= A^2(1 - \sin^2(\omega_0 t - \alpha)) \Rightarrow \sqrt{1 - \frac{x^2}{A^2}} = \sin^2$$

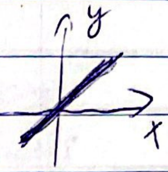
$$= \frac{B}{A}\left[x\cos(\delta) \pm A\sin(\delta)\sqrt{1 - \frac{x^2}{A^2}}\ \right]$$

$$\left(Ay - Bx\cos(\delta)\right)^2 = \left(\pm AB\sin\delta\sqrt{1 - \frac{x^2}{A^2}}\right)^2$$
$$A^2y^2 - 2ABxy\cos(\delta) + B^2x^2\cos^2\delta = A^2B^2\sin^2\delta\left(1 - \frac{x^2}{A^2}\right)$$
$$A^2y^2 - 2ABxy\cos(\delta) + B^2x^2\cos^2\delta = A^2B^2\sin^2\delta - x^2B^2\sin^2\delta$$
$$A^2y^2 - 2ABxy\cos(\delta) + B^2x^2(\cos^2\delta + \sin^2\delta) = A^2B^2\sin^2\delta$$

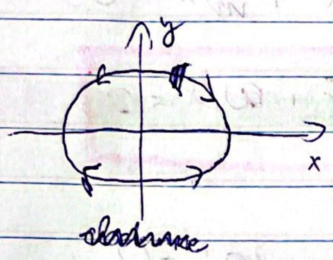
$$\Rightarrow \boxed{A^2y^2 + B^2x^2 - 2AB\,xy\cos\delta = A^2B^2\sin^2\delta}$$

## Special cases of $A^2y^2 + B^2x^2 - 2ABxy\cos(\delta) = A^2B^2\sin^2\delta$

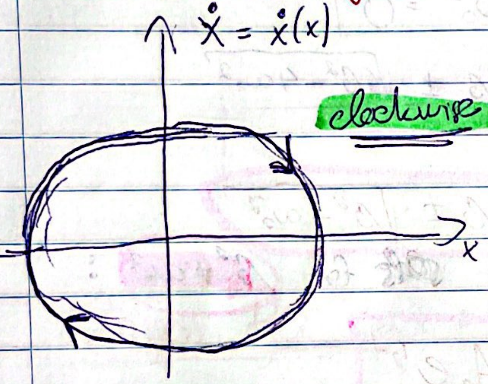
$\delta = 0\ (\alpha = \beta)$ :
$$A^2y^2 + B^2x^2 - 2ABxy = 0$$
$$(Ay - Bx)^2 = 0$$
$$Ay - Bx = 0 \Rightarrow \boxed{y = \frac{B}{A}x} \Rightarrow \text{line}$$




$\delta = \pm\frac{\pi}{2}$ :
$$A^2y^2 + B^2x^2 = A^2B^2$$
$$\boxed{\frac{y^2}{B^2} + \frac{x^2}{A^2} = 1} \Rightarrow \text{ellipse} \quad (\text{for } A = B : \text{circle})$$




## Phase diagram (10)




$$\hookrightarrow \boxed{\begin{aligned} x(t) &= A\cos(\omega_0 t - \varphi) \\ \dot{x}(t) &= -A\omega_0 \sin(\omega_0 t - \varphi) \end{aligned}}$$

$$x^2 = A^2\cos^2(\omega_0 t - \varphi)$$
$$\dot{x}^2 = A^2\omega_0^2 \sin^2(\omega_0 t - \varphi)$$
$$x^2 + \frac{\dot{x}^2}{\omega_0^2} = A^2(\cos^2(\omega t - \varphi) + \sin^2(\omega t - \varphi))$$
$$\Rightarrow \boxed{\frac{x^2}{A^2} + \frac{\dot{x}^2}{\omega_0^2 A^2} = 1} \Rightarrow \text{ellipse}$$

$$E = U + T$$
$$\frac{dU}{dx} = -F_x = kx \Rightarrow U = \frac{1}{2}kx^2 + C \hookrightarrow \boxed{\text{closed curves} \Rightarrow \text{Energy is conserved}}$$
$$T = \frac{1}{2}m\dot{x}^2$$
$$E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 \Rightarrow \frac{2E}{m} = x^2 + \frac{\dot{x}^2}{\omega_0^2} = \text{const.}$$
$$\Rightarrow 1 = \frac{x^2}{\frac{2E}{m}} + \frac{\dot{x}^2}{\omega_0^2 \frac{2E}{m}}$$

# Damped HO 1D

- damping usually function of velocity

$\Rightarrow \boxed{F_{damping}(v) = -\underset{\substack{\downarrow \\ \text{const.}}}{b}v}$ and $F_{spring} = -kx$

$\rightarrow m\ddot{x} = -kx - b\dot{x}$

Equation of motion: $\ddot{x} + \overset{2\beta}{\left(\frac{b}{m}\right)}\dot{x} + \overset{=\omega_0^2}{\left(\frac{k}{m}\right)}x = 0$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

Guess: $e^{rt}$, $r = \text{const.}$

$\Rightarrow r^2 e^{rt} + 2\beta r e^{rt} + \omega_0^2 e^{rt} = 0$

$e^{rt} \neq 0$

$\Rightarrow \boxed{r^2 + 2\beta r + \omega_0^2 = 0}$

$\hookrightarrow r_{1/2} = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$

$\Rightarrow \boxed{r_{1/2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}}$

leads to two lin. indep. ~~sols~~ for $\beta^2 \neq \omega_0^2$:

$$\boxed{x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}}$$

$\hookrightarrow$ 3 cases which can occur:

① **underdamping** $\beta < \omega_0$ – weak damping force

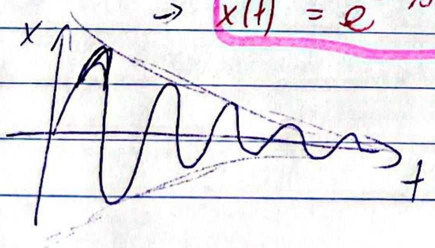
$r_{1/2} = -\beta \pm i\underbrace{\sqrt{|\omega_0^2 - \beta^2|}}_{\text{let}}$  $\omega_1^2 \equiv \omega_0^2 - \beta^2 > 0$

$\Rightarrow r_{1/2} = -\beta \pm i\omega_1$

$\Rightarrow e^{r_1 t} = e^{-\beta t} e^{i\omega_1 t}$

$\Rightarrow x(t) = A_1 e^{-\beta t} e^{i\omega_1 t} + A_2 e^{-\beta t} e^{-i\omega_1 t}$

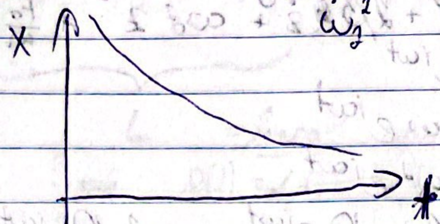
$\Rightarrow \boxed{x(t) = e^{-\beta t}\left(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right)}$




decaying oscillations (envelope $\propto e^{-\beta t}$)

② **overdamping** $\boxed{\beta > \omega_0}$ – strong damping force

$r_{1,2} = -\beta \pm \sqrt{\underbrace{\beta^2 - \omega_0^2}_{\omega_2^2}}$ ⇒ only exponential decay, without oscillations



$x(t) = e^{-\beta t}\left(A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right)$

③ **critical damping** $\boxed{\beta = \omega_0}$

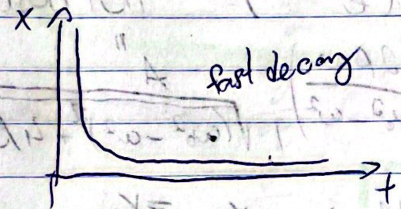
$r = -\beta$ ⇒ only one solution but we need two lin. indep. for general sol.

⇒ as a second solution: $te^{rt}$

⇒ $\boxed{x(t) = A_1 e^{rt} + A_2 te^{rt}}$

↳ exp stronger than lin. $t$

⇒ still decay in time




**Driven HO 1D:** $F_x = -kx$ still: $\omega_0 = \sqrt{\frac{k}{m}}$ natural freq.

$F_{damp} = -bv$

$F_{driving} = F_0 \cos(\omega t)$ ($\omega \neq \omega_0$)

$F_x + F_{damp} + F_{driving} = ma = m\ddot{x}$

$-kx - b\dot{x} + F_0 \cos(\omega t) = m\ddot{x}$

⇒ $\ddot{x} + \underbrace{\frac{b}{m}}_{2\beta}\dot{x} + \underbrace{\frac{k}{m}}_{\omega_0^2}x = \underbrace{\frac{F_0}{m}}_{A}\cos(\omega t)$

Eq. of motion ⇒ $\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos(\omega t)}$ non-homogeneous

**general solution:** $x_{gen} = x_{homogeneous} + x_{particular}$ of non-homo.

same as we had before

## Finding particular solution

17/11 let $z(t) = x + iy$, $i(\ddot{y} + 2\beta\dot{y} + \omega_0^2 y) = i\frac{F_0}{m}\sin(\omega t)$
$\hookrightarrow \mathrm{Re}(z) = x$ $\quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$

$$(\ddot{x} + i\ddot{y}) + 2\beta(\dot{x} + i\dot{y}) + \omega_0^2(x+iy) = \frac{F_0}{m}(\cos(\omega t) + i\sin(\omega t))$$
$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t} \equiv Ae^{i\omega t}$$

guess: $z_p(t) = De^{i\omega t}$
$$\dot{z}_p = Di\omega e^{i\omega t}$$
$$\ddot{z}_p = -D\omega^2 e^{i\omega t}$$
$$-D\omega^2 e^{i\omega t} + 2\beta i\omega De^{i\omega t} + \omega_0^2 De^{i\omega t} = Ae^{i\omega t}$$
$$-D\omega^2 + i\omega 2\beta D + \omega_0^2 D = A$$
$$\Rightarrow D = \frac{A}{\omega_0^2 - \omega^2 + i\omega 2\beta} = |D|e^{i\omega t}$$
$$x = \mathrm{Re}(z) = \mathrm{Re}\left(\frac{A}{\omega_0^2 - \omega^2 + i\omega 2\beta}e^{i\omega t}\right) =$$
$$= \mathrm{Re}\left((\omega_0^2 - \omega^2 + i\omega 2\beta)^{-1}\cos(\omega t) + i\sin(\omega t)\right)$$
$$= \|D\|\,\mathrm{Re}(e^{i(\omega t + \delta)}) = |D|\cos(\omega t + \delta)$$
$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right), \quad \overset{\shortparallel}{\frac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}}$$

$\boxed{x_p = \text{long-term solution}}$ ; $x_c = x_{homo}$ decays fast with time

$$\Rightarrow \boxed{x_p(t) = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}\cos\left(\omega t + \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)\right)}$$

**Resonance** – for a particular $\omega$, amplitude of oscillations becomes very large

$\Rightarrow$ at resonance, maximum $D$:
$$\frac{d|D|}{d\omega} = 0 \Rightarrow \boxed{\omega = \omega_R := \sqrt{\omega_0^2 - 2\beta^2}}$$
resonance frequency

when $\omega_0 \gg \beta$ & nearly resonance:
Taylor expand $D$ in terms of $\frac{\beta}{\omega_0}$ (small):
$$\boxed{|D|\Big|_{\omega=\omega_R} \cong \frac{A}{2\beta\omega_0}\left(1 + \frac{\beta^2}{2\omega_0^2} + \dots\right)}$$

If there is no damping ($\beta = 0$) $\Rightarrow$ $D \to \infty$

# Green's Function & Fourier series

$$\underbrace{\left(\frac{d^2}{dt^2} + a\frac{d}{dt} + b\right)}_{L} x(t) = f(t)$$

$L$ = linear operator $\Rightarrow$ use principle of superposition

principle of superposition $\left\{ f(t) = \alpha_1 f_1(t) + f_2(t)\alpha_2 \right.$, $f_1(t) = Lx_1(t)$, $f_2(t) = Lx_2(t)$ $\Rightarrow$ $\boxed{x(t) = \alpha_1 x_1(t) + x_2(t)\alpha_2}$ solves $Lx = f(t)$

$$\Rightarrow f(t) = \sum_{n=1}^{N} \alpha_n f_n(t) \Rightarrow x(t) = \sum_{n=1}^{N} \alpha_n x_n(t)$$

with $Lx_n(t) = f_n(t)$

- if we have periodic functions $F(t)$ (with period $\tau = \frac{2\pi}{\omega}$), we can ~~use~~ express them as their Fourier series

$$\boxed{F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]}$$

with
$$\boxed{\begin{aligned} a_n &= \frac{2}{\tau}\int_0^{\tau} \cos(n\omega t')\, F(t')\, dt' \\ b_n &= \frac{2}{\tau}\int_0^{\tau} \sin(n\omega t')\, F(t')\, dt' \end{aligned}}$$

useful trig identities: $\cos(\phi + \theta) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$\int_0^{\tau} \cos(n\omega t)\cos(m\omega t)\, dt = \frac{\tau}{2}\delta_{nm}$$

$$\int_0^{\tau} \sin(n\omega t)\sin(m\omega t)\, dt = \frac{\tau}{2}\delta_{nm}$$

## Green's function method

- Green's function $G$ is s.t. $\boxed{LG = \delta(t-t')}$ ↳ Dirac delta distribution

$$\Rightarrow \boxed{x(t) = \int_{-\infty}^{\infty} G(t,t')\, \frac{F(t')}{m}\, dt'}$$

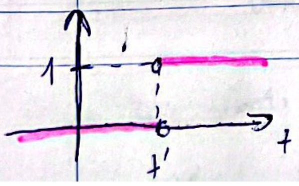
because $L_t x(t) = \int_{-\infty}^{\infty} \underbrace{(L_t G(t,t'))}_{\delta(t-t')} \frac{F(t')}{m}\, dt'$

$$\int_{-\infty}^{\infty} f(t')\,\delta(t-t')\,dt' = f(t)$$

because $G$ is the only $t$-dep. function

$$= \int_{-\infty}^{\infty} \delta(t-t')\, \frac{F(t')}{m}\, dt'$$
$$= F(t)/m \checkmark$$

**HO:** For $\omega_0 \neq \beta$:




$$G(t,t') = \begin{cases} \theta(t-t')\,\dfrac{e^{-\beta(t-t')}}{\sqrt{\omega_0^2-\beta^2}}\,\sin\left[\sqrt{\omega_0^2-\beta^2}\,(t-t')\right], & \omega_0 > \beta \\ \theta(t-t')\,\dfrac{e^{-\beta(t-t')}}{\sqrt{\omega_0^2-\beta^2}}\,\sinh\left[\sqrt{\beta^2-\omega_0^2}\,(t-t')\right], & \omega_0 < \beta \end{cases}$$

$$= \theta(t-t')\, e^{-\beta(t-t')}\, \frac{1}{2i}(\omega_0 - \beta^2)^{-1/2}\left(e^{i\sqrt{\omega_0^2-\beta^2}(t-t')} - e^{-i\sqrt{\omega_0^2-\beta^2}(t-t')}\right)$$

★ $F(t) = \begin{cases} D, & t>0 \\ 0, & t<0 \end{cases} = D\,\theta(t)$, forcing function of HO

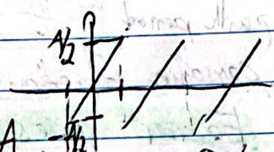
$\Rightarrow x(t) = \int_{-\infty}^{\infty} G(t,t') \frac{F(t')}{m} dt' = \int_{0}^{\infty} G(t,t') \frac{D}{m} dt'$

use $G(t,t')$ from previous page

Newton laws : $\vec{F} = m\vec{a}$ $\Rightarrow$ old

Now: new approach : variational calculus

Example 3.6
- saw tooth



$F(t) = A\frac{t}{\tau} = \frac{\omega A}{2\pi} t$, $-\tau/2 < t < \tau/2$
$\tau = \frac{2\pi}{\omega}$

odd $\Rightarrow a_n = 0$

$b_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} \sin(n\omega t)\, A \frac{\omega}{2\pi} t \, dt =$

$= \frac{\omega 2}{2\pi} A \frac{\omega}{2\pi} 2 \int_0^{\tau/2} t \sin(n\omega t)\, dt =$

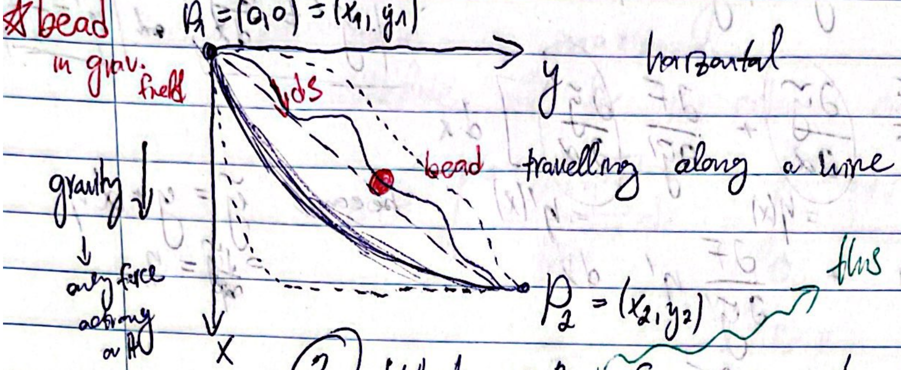
$= A \frac{\omega^2}{\pi^2} \left[ -t \frac{\cos(n\omega t)}{n\omega} \Big|_0^{\tau/2} + \int_0^{\tau/2} \frac{\cos(n\omega t)}{n\omega} dt \right] =$

$= A\,\omega^2/\pi^2 \left[ \frac{\sin(n\omega t)}{n^2\omega^2} \Big|_0^{\tau/2} - \frac{\tau}{2} \frac{\cos(n\omega \frac{\pi}{\omega})}{n\omega} \right] =$

$= A \frac{\omega^2}{\pi^2} \left[ \frac{\sin(n\pi)}{n^2\omega^2} - \frac{\tau \cos(n\pi)}{2n\omega} \right] = -A \frac{\omega^2}{\pi^2} \frac{(-1)^n}{n\omega} \frac{2\pi}{2\omega}$

$= -A \frac{(-1)^n}{\pi n} = \frac{A}{\pi n} (-1)^{n+1}$

# Variational principle




this path will be described by $y(x)$

(?) What path for wire do we want to choose such that the $\underbrace{\text{travel time}}_{\Delta T}$ $P_1 \to P_2$ is minimal?

$$\Delta T = \int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{dt}{dx}\, dx = \int_{x_1}^{x_2} \frac{dt}{ds}\frac{ds}{dx}\, dx = *$$

$E = T + U = \frac{1}{2}mv^2 - mgx$

let initially: $v = 0$ at $P_1 : x = 0$

$\Rightarrow$ initially $\boxed{E = 0}$ but energy is conserved

$\Rightarrow E = 0$ always $\Rightarrow \frac{1}{2}mv^2 - mgx = 0$

$\Rightarrow v^2 = 2gx \Rightarrow v = \sqrt{2gx}$

$$\boxed{\frac{dt}{ds} = \frac{1}{v}} \Leftarrow v = \frac{d|\vec{s}|}{dt} = \frac{ds}{dt} = \frac{d}{dt}\sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + (y')^2}$$

$\Rightarrow \frac{ds}{dx} = \sqrt{1 + (y')^2}$ $\quad \frac{dy}{dx} = y'$

$$* \; \Delta T = \int_{x_1}^{x_2} \frac{1}{v}\sqrt{1 + (y')^2}\, dx = \int_{x_1}^{x_2} \frac{1}{\sqrt{2gx}}\sqrt{1 + y'^2}\, dx =$$

$$\boxed{\Delta T = \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{2gx}}\, dx}$$ travel time along path $y(x)$

**Functional** $\boxed{J[y] = \int_{x_1}^{x_2} F[y, y'; x]\, dx}$ (here $\Delta T = J$)

↳ function of functions; $x$ = function of $x$

- arbitrary function between $P_1$ and $P_2$: $\tilde{y}(x) = y(x) + \alpha\, \eta(x)$ ($\alpha$ = const., $\eta$ = arbitrary continuous function)

written parametrically in terms of $y(x)$.

When $\alpha = 0 \Rightarrow \tilde{y}(x) = y(x)$ = minimal travel time path

we want: $\boxed{\left.\frac{\partial J[\tilde{y}]}{\partial \alpha}\right|_{\alpha=0} = 0}$, here $\left.\frac{\partial \Delta T}{\partial \alpha}\right|_{\alpha=0} = 0$

Integration by parts: $\int_{x_1}^{x_2} f \cdot g' = \int_{x_1}^{x_2} \frac{d}{dx}(fg) - f'g \, dx = f \cdot g \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} f' \cdot g \, dx$

$$J[\tilde{y}] = \int F[\tilde{y}(x), \tilde{y}'(x), x] \, dx \qquad \boxed{\tilde{y}' = \frac{d\tilde{y}}{dx}} \text{ and } \boxed{\eta' = \frac{d\eta}{dx}}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \Big[ \frac{\partial F}{\partial \tilde{y}} \underbrace{\frac{\partial \tilde{y}}{\partial \alpha}}_{=\eta(x)} + \frac{\partial F}{\partial \tilde{y}'} \underbrace{\frac{\partial \tilde{y}'}{\partial \alpha}}_{=\eta'(x)} \Big] dx$$

because $\tilde{y} = y + \alpha\eta \Rightarrow \frac{\partial \tilde{y}}{\partial \alpha} = \eta$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial \tilde{y}} \eta + \frac{\partial F}{\partial \tilde{y}'} \eta' \, dx$$

integrate by parts, $\eta(x_1) = 0 = \eta(x_2)$

$$= \int_{x_1}^{x_2} \underbrace{\left( \frac{\partial F}{\partial \tilde{y}} - \frac{d}{dx} \frac{\partial F}{\partial \tilde{y}'} \right)}_{=0} \eta(x) \, dx = 0$$

$\eta(x)$: arbitrary function, so if we want $\int = 0$, we need the other part $= 0$

$$\Rightarrow \boxed{\frac{\partial F}{\partial \tilde{y}} - \frac{d}{dx} \frac{\partial F}{\partial \tilde{y}'} = 0}$$

**Euler's Equation**

useful when F indep. of y — but holds always

Derivative of $\tilde{y}$

21/11 ... continuing problem with a bead: $\underbrace{\Delta T}_{J} = \int_{x_1}^{x_2} \underbrace{\frac{\sqrt{1+y'^2}}{\sqrt{2gx}}}_{F} dx$

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \qquad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}\sqrt{x}}$$

$\frac{\partial F}{\partial y} = 0$ because F doesn't explicitly depend on y

$$\Rightarrow \frac{d}{dx} \frac{\partial F}{\partial y'} = 0$$

$$\Rightarrow \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{x}\sqrt{1+y'^2}} = \text{const.} = c = \frac{1}{\sqrt{2a}}$$

$$\left[ \frac{\frac{\text{length}}{\text{length}}}{\sqrt{\text{length}}} \right] = [\text{length}^{-1/2}]$$

square both sides

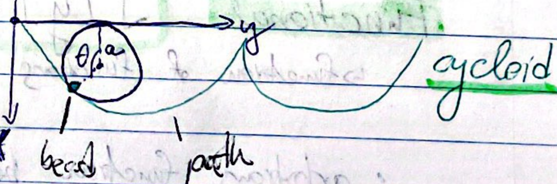
$$2a\, y'^2 = x(1+y'^2)$$

$$y'^2 - \frac{x y'^2}{2a} = \frac{x}{2a}$$

$$y'^2 = \frac{x}{2a(1-\frac{x}{2a})} = \frac{x}{2a-x}$$

$$\frac{dy}{dx} = y' = \sqrt{\frac{x}{2a-x}}$$

$$y = \int \sqrt{\frac{x}{2a-x}} \, dx = \int \frac{x \, dx}{\sqrt{2ax - x^2}}$$




$\hookrightarrow x = a(1-\cos\theta)$, $dx = a \sin\theta \, d\theta$

$$y = \int \frac{a^2(1-\cos\theta)\sin\theta \, d\theta}{\sqrt{2a^2(1-\cos\theta) - a^2(1-\cos\theta)^2}} =$$

$$= a \int \frac{(1-\cos\theta)\sin\theta \, d\theta}{\sqrt{2(1-\cos\theta) - \sin^2\theta}} = a \int 1 - \cos\theta \, d\theta = a(\theta - \sin\theta)$$



Scanned with CamScanner

$\Rightarrow$ the extremum is $y = a(\theta - \sin\theta)$

(?) minimum or maximum

$\hookrightarrow$ calculate $\delta T$ for this extreme curve and for one other curve

$$\delta T = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}\,dx = \int_0^{\pi} \frac{\sqrt{1+a^2(1-\cos\theta)^2}}{\sqrt{2g\,a(1-\cos\theta)}}\, a\sin\theta\, d\theta$$

$x_1 = 0 = \theta_1$

$x_2 = 2a \Rightarrow \theta_2 = \pi$

$$\delta T = \frac{1}{\sqrt{2g}} \int_0^{\pi} \sqrt{\frac{1}{a(1-\cos\theta)} + a(1-\cos\theta)}\; a\sin\theta\, d\theta = \dots = \sqrt{\frac{a}{g}}\,\pi$$

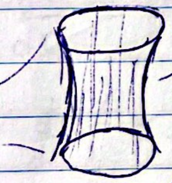
$\Rightarrow$ compare to arbitrary curve, e.g. $y = x$:

$$\delta T = \int_{x_1}^{x_2} \frac{\sqrt{1+1^2}}{\sqrt{2gx}}\,dx = \int_0^{2a} \sqrt{\frac{2}{2gx}}\,dx = \frac{1}{\sqrt{g}} \int_0^{2a} x^{-1/2}\,dx =$$

$$= \frac{1}{\sqrt{g}}\, 2\sqrt{x}\,\Big|_0^{2a} = \frac{2\sqrt{2a}}{\sqrt{g}} = 2\sqrt{2}\sqrt{\frac{a}{g}} > \pi\sqrt{\frac{a}{g}}$$

$\Rightarrow$ found extremum is a minimum

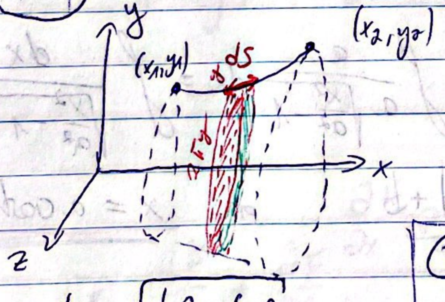
## ☆ Soap film problem



metal rings

soap film $\Rightarrow$ for lowest energy, minimise surface tension

$\Rightarrow$ minimise surface




- take a line on xy-plane, then rotate about the x-axis,

$\Rightarrow$ rotational (cyl. sym.) surface

(?) find a line between $(x_1, y_1)$ and $(x_2, y_2)$ which gives us minimal surface upon rotation about x-axis

$ds = \sqrt{dx^2 + dy^2}$

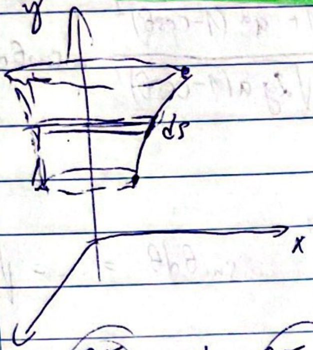
$dA = 2\pi y\, ds$

$$A = \int dA = \int 2\pi y \sqrt{dx^2+dy^2} = \int_{x_1}^{x_2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi \int_{x_1}^{x_2} \underbrace{y\sqrt{1+y'^2}}_{F}\,dx$$

$$\underbrace{\frac{\partial F}{\partial y}}_{\sqrt{1+y'^2}} - \frac{d}{dx}\underbrace{\frac{\partial F}{\partial y'}}_{\frac{y \cdot y'}{\sqrt{1+y'^2}}} = 0 \Rightarrow \boxed{\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0}$$ Euler eq.

!! complicated

$\Rightarrow$ complicated, so we want to choose a different approach $\Rightarrow$ change coordinate system
$\Rightarrow$ rotate instead about $y$-axis and keep $x$ as indep. variable



$\Rightarrow ds = \sqrt{dx^2 + dy^2}$

$dA = 2\pi x\, ds = 2\pi x\, dx \sqrt{1+y'^2}$

$\Rightarrow A = \int dA = 2\pi \int_{x_1}^{x_2} \underbrace{x\sqrt{1+y'^2}}_{F}\, dx$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$\frac{\partial F}{\partial y} = 0$ ☺, $\frac{\partial F}{\partial y'} = \frac{xy'}{\sqrt{1+y'^2}}$ $\Rightarrow \frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^2}}\right) = 0$

$\Downarrow$

$\frac{xy'}{\sqrt{1+y'^2}} = \text{const.} = a$ [length]

$x^2 y'^2 = a^2(1+y'^2) = a^2 + a^2 y'^2$

$x^2 y'^2 - a^2 y'^2 = a^2$

$y'^2(x^2 - a^2) = a^2$

$y'^2 = \frac{a^2}{x^2 - a^2}$

$\frac{dy}{dx} = y' = \frac{a}{\sqrt{x^2-a^2}}$

$\Rightarrow y = \int \frac{a}{\sqrt{x^2-a^2}}\, dx = \int \frac{a}{a\sqrt{\frac{x^2}{a^2}-1}}\, dx = \int \frac{dx}{\sqrt{\frac{x^2}{a^2}-1}}$

$\Rightarrow \underline{\underline{y = a\cosh^{-1}\left(\frac{x}{a}\right) + b}}$ or $\underline{\underline{x = a\cosh\left(\frac{y-b}{a}\right)}}$

## Multidimensional & independent $y$'s

- we can have $F[y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n'; x]$
  $\Rightarrow$ multiple dependent functions, one indep. variable

  e.g. $y(x), z(x)$ $\Rightarrow$ line in 3D space

  $\Rightarrow$ we want to parametrise: $\tilde{y}_i(x, \alpha_i) = y_i(x) + \alpha\eta_i(x)$

$\Rightarrow$ again $\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \sum_{i=1}^{n} \left(\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'}\right)\eta_i\, dx = 0 \Rightarrow$ Euler Eq. for each $i = 1, \dots, n$

because all funs are indep.

## Second formulation of the Euler Eq.

⇒ useful when Functional integrand $F$ is independent of $x$ (but can explicitly depend on $y$)

dep. var., derivative, independent variable

$$\frac{dF[y, y'; x]}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dx}}_{y'} + \frac{\partial F}{\partial y'}\underbrace{\frac{dy'}{dx}}_{=y''}$$

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$\Rightarrow y''\frac{\partial F}{\partial y'} = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

substitute into first equation:

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$= \frac{\partial F}{\partial x} + \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) + y'\underbrace{\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]}_{=0 \text{ from Euler's eq.}}$$

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}$$

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial F}{\partial x} - \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0}$$

2nd form of Euler's eq. useful when $F$ indep of $x$ (but holds always)

$\frac{\partial F}{\partial x} = 0 \Leftarrow$ explicitly at least

$$\frac{dF}{dx} \neq \frac{\partial F}{\partial x}$$

all x-dependence; only explicit x-dep. (not containing $y(x)$)

★ Soap film – 1st formulation: $A = 2\pi\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$

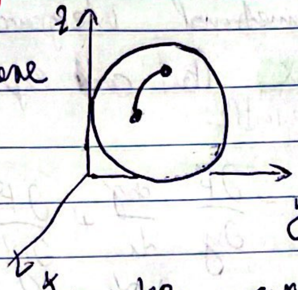
$\Rightarrow \frac{\partial F}{\partial x} = 0$

$$F - y'\frac{\partial F}{\partial y'} = y\sqrt{1+y'^2} - y'y\frac{y'}{\sqrt{1+y'^2}} = \text{const.}$$

⇒ same solution as before

# Lagrange multipliers

- sphere




2 dimensional surface

x – indep., curve on sphere: $\begin{cases} y(x) \\ z(x) \end{cases}$

but constraint to surface of the sphere!

$\Rightarrow$ y and z are no longer independent

$\Rightarrow x^2+y^2+z^2=\rho^2$

$\Rightarrow g(y,z;x) = x^2+y^2+z^2-\rho^2 = 0$

? Euler equations when y & z aren't independent?

$$J = \int_{x_1}^{x_2} F[y(x), z(x), y', z'; x]\, dx$$

$$\frac{\partial J}{\partial \alpha} = 0 \qquad \tilde{y}(x,\alpha) = y(x) + \alpha \eta_y(x)$$
$$\tilde{z}(x,\alpha) = z(x) + \alpha \eta_z(x)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \alpha} dx = \ldots =$$

chain rule then integrate by parts to remove $\eta'$

not independent anymore

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta_y + \left(\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'}\right)\eta_z\right] dx = 0$$

Now we use the constraint: $\frac{dg}{d\alpha} = \left(\frac{\partial g}{\partial y}\underbrace{\frac{\partial \tilde{y}}{\partial \alpha}}_{\eta_y} + \frac{\partial g}{\partial z}\underbrace{\frac{\partial \tilde{z}}{\partial \alpha}}_{\eta_z}\right) = 0$

$$\eta_z = -\frac{\partial g}{\partial y}\eta_y\left(\frac{\partial g}{\partial z}\right)^{-1}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta_y - \underbrace{\left(\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'}\right)\frac{\partial g}{\partial y}\left(\frac{\partial g}{\partial z}\right)^{-1}}_{=-\lambda(x)}\eta_y\right] dx = 0$$

for arbitrary $\eta_y(x)$

$$\Rightarrow \boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{\partial g}{\partial y}\lambda(x) = 0}$$

$$+ \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial y}$$

if we have m constraints

= Lagrange multiplier

Constraint eq. $\quad g(y,z;x)=0$

Euler eq. for $y$: $\quad \dfrac{\partial F}{\partial y}-\dfrac{d}{dx}\dfrac{\partial F}{\partial y'}+\dfrac{\partial g}{\partial y}\lambda(x)=0$

Euler eq. for $z$: $\quad \dfrac{\partial F}{\partial z}-\dfrac{d}{dx}\dfrac{\partial F}{\partial z'}+\dfrac{\partial g}{\partial z}\lambda(x)=0$

3 equations,

3 unknown functions of $x$

- $y(x)$
- $z(x)$
- $\lambda(x)$

We can verify that for both $y$ and $z$:

$$-\lambda(x)=\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}\right)\left(\frac{\partial g}{\partial y}\right)^{-1}=\left(\frac{\partial F}{\partial z}-\frac{d}{dx}\frac{\partial F}{\partial z'}\right)\left(\frac{\partial g}{\partial z}\right)^{-1}$$

# Hamiltonian Mechanics

↳ using Lagrangian: $S=\int L$ functional to extremise

↳ trying to formulate laws of motion

↳ using variational techniques to derive laws of physics

## Fermat's principle

- postulate
- light takes path of minimal travel time $\delta t$

$P_1 \to ds \to P_2$ in 3D $\quad ds^2=dx^2+dy^2+dz^2$

$$\delta t=\int_{P_1}^{P_2}dt=\int_{P_1}^{P_2}\frac{ds}{c}$$

parametrization of curve $\begin{cases}x=\alpha\\ y=f(\alpha)\\ z=g(\alpha)\end{cases}$

$$ds=\sqrt{\left(\frac{dx}{d\alpha}\right)^2+\left(\frac{dy}{d\alpha}\right)^2+\left(\frac{dz}{d\alpha}\right)^2}\,d\alpha$$

$$ds=\sqrt{1+f'^2+g'^2}\,d\alpha$$

$$\delta t=\frac{1}{c}\int_{P_1}^{P_2}\underbrace{\sqrt{1+f'^2+g'^2}}_{F}\,d\alpha$$

$$\begin{cases}\dfrac{\partial F}{\partial f}-\dfrac{d}{d\alpha}\dfrac{\partial F}{\partial f'}=0\\[2ex] \dfrac{\partial F}{\partial g}-\dfrac{d}{d\alpha}\dfrac{\partial F}{\partial g'}=0\end{cases}$$

$\dfrac{\partial F}{\partial f}=0$, $\quad \dfrac{\partial F}{\partial f'}=\dfrac{f'}{\sqrt{1+f'^2+g'^2}}=a_f$

$\dfrac{\partial F}{\partial g}=0$, $\quad \dfrac{\partial F}{\partial g'}=\dfrac{g'}{\sqrt{1+f'^2+g'^2}}=a_g$

$\Rightarrow$ solve $\dfrac{f'}{\sqrt{1+f'^2+g'^2}} = a_f$ and $\dfrac{g'}{\sqrt{1+f'^2+g'^2}} = a_g$

$\Rightarrow$ get : $\begin{cases} f = \alpha A + B \\ g = \alpha C + D \end{cases}$

$\Rightarrow$ light moves in straight lines in homog. isotr. medium

## Classical system

– only for conservative (otherwise we can't use potential energy e.g. Lorentz force)

Q: Is there J to use euler's eq's?

**Lagrangian** $\boxed{L \equiv T - U}$

$T$: kinetic energy, $U$: potential energy

$\boxed{S = \int L \, dt}$

$S$: **action**

$\Rightarrow$ by extremising action, we get equations of motion (for conservative forces) same as from Newton's

## Free particle in 1D

Newton: $F = 0 = ma = m\ddot{x} \Rightarrow \ddot{x} = 0 \Rightarrow \dot{x} = \text{const} \Rightarrow x = \text{line}$

Lagrangian: $L = T - \underset{\substack{\parallel \\ 0 \text{ for free particle}}}{U} = \frac{1}{2} m \dot{x}^2$ : ~~ke~~

$\Rightarrow L = L(\underbrace{x, \dot{x}}_{\text{dep}}; \underbrace{t}_{\text{indep}})$

$\Rightarrow$ euler's eq: $\underbrace{\dfrac{\partial L}{\partial x}}_{0} - \dfrac{d}{dt} \underbrace{\dfrac{\partial L}{\partial \dot{x}}}_{m\dot{x}} = 0 \Rightarrow m\ddot{x} = 0$

$\cancel{m}\dot{x} = \text{const}$

✓

... Add force $\star$ $F = -\frac{\partial U}{\partial x}$

$\star$ $L = T - U = \frac{1}{2} m \dot{x}^2 - U(x)$

$$\underbrace{\frac{\partial L}{\partial x}}_{-\frac{\partial U}{\partial x}} - \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{x}}}_{m\dot{x}} = 0 \quad , \quad \frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$$

$$-\frac{\partial U}{\partial x} - m\ddot{x} = 0 \Rightarrow m\ddot{x} = -\frac{\partial U}{\partial x} = F \Rightarrow ma = F \checkmark$$

Newton

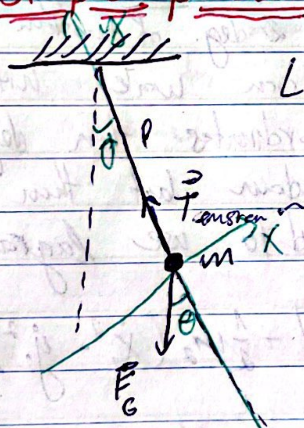
... in 3D: $S = \int L[x, y, z, \dot{x}, \dot{y}, \dot{z}; t]\, dt$

$\vec{r}(t) = (x(t), y(t), z(t))$

$$L = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

$$\begin{cases} \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \\ \vdots \end{cases}$$

Simple pendulum



Newton: $\Rightarrow$ $\ddot{\theta} + \frac{g}{l} \sin\theta = 0$

Lag.-Euler:

$L = \cancel{K}T - U$

Tension $\rightsquigarrow$ reducing number of degrees of freedom

$$\left.\begin{array}{l} F_x = -mg\sin\theta = ma_x \\ F_y = -mg\cos\theta + T = 0 \quad \text{~~From~~} \\ \quad \hookrightarrow a_y = 0 \\ a_x = \frac{d}{dt} v_x = \frac{d}{dt} (l\dot{\theta}) = l\ddot{\theta} \\ ml\ddot{\theta} + m \frac{g}{l} \sin\theta = 0 \\ \ddot{\theta} + \frac{g}{l} \sin\theta = 0 \end{array}\right\} \text{Newtonian}$$

$\hookrightarrow$ in 2D - 2 coordinates

$\hookrightarrow x^2 + y^2 = l^2$ - 1 constraint

$\Downarrow$

1 coordinate

$L = \cancel{K}T - U$ : $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m l^2 \dot{\theta}^2$

$U = mg\, l(1 - \cos\theta)$ $\hookleftarrow$ zero when bottom of pendulum

$$\Rightarrow L = \frac{m}{2} l^2 \dot{\theta}^2 - mgl(1 - \cos\theta) = L(\theta, \dot{\theta}; t)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0$$

$$\frac{\partial L}{\partial t} - \frac{d}{dt}\left(L - \dot\theta \frac{\partial L}{\partial \dot\theta}\right) = 0 \quad \text{vs.} \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 0$$

$$-\frac{d}{dt}\left(\frac{1}{2} m \dot\theta l^2 - \dot\theta\, m \dot\theta l^2\right) = 0 \qquad -mgl\sin\theta \quad \frac{d}{dt}(m\dot\theta l^2) = m\ddot\theta l^2$$

$$\dot\theta^2 l^2 - \frac{1}{2}\dot\theta l^2 = \text{const.} = C \qquad mgl\sin\theta + m\ddot\theta l^2 = 0$$

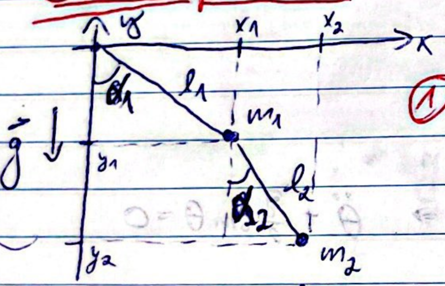
$$\dot\theta^2 - \frac{1}{2}\dot\theta = \text{const.} = -C \Rightarrow \ddot\theta + \frac{g}{l}\sin\theta = 0 \checkmark$$

Same as Newton

$$\dot\theta^2 - \frac{1}{2}\dot\theta + C = 0$$

$$\dot\theta = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4C}}{2}$$

## Double pendulum



! correct number of variables is important to getting correct answer

① degrees of freedom

2 degrees of freedom

- $2 \times$ 2 degrees of freedom of movement in plane
- 2 constraints by $l_1, l_2$

$\Rightarrow$ 2 deg. of freedom

$\hookrightarrow$ we can work with more coordinates than degrees of freedom but then we need to use lagrange multipliers

use angles

$x_1 = l_1 \sin\alpha_1$

$y_1 = -l_1 \cos\alpha_1$

$x_2 = l_1\sin\alpha_1 + l_2\sin\alpha_2$

$y_2 = -l_1\cos\alpha_1 - l_2\cos\alpha_2$

① $x_1^2 + y_1^2 = l_1^2$

② $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$

② kinetic energy

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (\dot x_1^2 + \dot y_1^2) + \frac{1}{2} m_2 (\dot x_2^2 + \dot y_2^2)$$

$$\dot x_1 = l_1 \cos(\alpha_1)\,\dot\alpha_1$$

$$\dot y_1 = l_1 \sin(\alpha_1)\,\dot\alpha_1$$

$$\dot x_2 = l_1 \cos(\alpha_1)\,\dot\alpha_1 + l_2 \cos(\alpha_2)\,\dot\alpha_2$$

$$\dot y_2 = l_1 \sin(\alpha_1)\,\dot\alpha_1 + l_2 \sin(\alpha_2)\,\dot\alpha_2$$

$$T = \frac{m_1}{2} l_1^2 (\sin^2\alpha_1 + \cos^2\alpha_1)\dot\alpha_1^2 + \frac{m_2}{2}\big(l_1^2\dot\alpha_1^2\cos^2\alpha_1 + l_2^2\dot\alpha_2^2\cos^2\alpha_2 + 2l_1l_2\dot\alpha_1\dot\alpha_2\cos(\alpha_1)\cos(\alpha_2) + l_1^2\dot\alpha_1^2\sin^2\alpha_1 + l_2^2\dot\alpha_2^2\sin^2\alpha_2 + 2l_1l_2\dot\alpha_1\dot\alpha_2\sin(\alpha_1)\sin(\alpha_2)\big)$$

$$= \frac{m_1}{2} l_1^2\dot\alpha_1^2 + \frac{m_2}{2}\big(l_1^2\dot\alpha_1^2 + l_2^2\dot\alpha_2^2 + 2l_1l_2\dot\alpha_1\dot\alpha_2[\cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2)$$

$$\Rightarrow \boxed{T = l_1^2 \dot{\alpha}_1^2 \left(\frac{m_1}{2} + \frac{m_2}{2}\right) + l_2^2 \dot{\alpha}_2^2 \frac{m_2}{2} + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2)}$$

(3) Potential E: potential energy

$$U = U_1 + U_2 = m_1 g y_1 + m_2 g y_2 = -m_1 g l_1 \cos\alpha_1 - m_2 g (l_1 \cos\alpha_1 + l_2 \cos\alpha_2) =$$

$$\boxed{U = -m_1 g l_1 \cos\alpha_1 - m_2 g (l_1 \cos\alpha_1 + l_2 \cos\alpha_2)}$$

(4) Lagrangian $L = T - U = L(\alpha_1, \alpha_2, \dot{\alpha}_1, \dot{\alpha}_2 ; t)$

$$\boxed{\frac{\partial L}{\partial \alpha_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}_i} = 0}$$

(5) Euler equations

$\frac{\partial L}{\partial \dot{\alpha}_1}$: $l_1^2 \dot{\alpha}_1 (m_1 + m_2) + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2)$

$\frac{d}{dt} \hookrightarrow l_1^2 \ddot{\alpha}_1 (m_1 + m_2) + m_2 l_1 l_2 (\ddot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)(-\dot{\alpha}_1 + \dot{\alpha}_2))$

$\frac{\partial L}{\partial \alpha_1}$: $-m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) - (m_1 g l_1 \sin(\alpha_1) + m_2 g l_1 \sin \alpha_1)$

$$-m_2 l_1 l_2 \sin(\alpha_1 - \alpha_2) - g l_1 \sin(\alpha_1)(m_1 + m_2) - l_1^2 \ddot{\alpha}_1 (m_1 + m_2) - m_2 l_1 l_2 \cos(\alpha_1 - \alpha_2) \ddot{\alpha}_2 - m_2 l_1 l_2 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)(-\dot{\alpha}_1 + \dot{\alpha}_2) = 0$$

... an same for $\alpha_2$. Then solve this coupled system of ODEs. (6) Solve ODEs

# Generalised coordinates

- not just cartesian, can be anything

$$x_{\alpha, i} = x_{\alpha, i}(q_j ; t)$$

$\alpha$: index of / # point masses running up to the # point masses

$i = 1,2,3$: cartesian coordinates index $1 = x$, $2 = y$, $3 = z$

$j = 1, \dots, n$: index of generalised coordinate, running up to # degrees of freedom

$n$ = # point masses

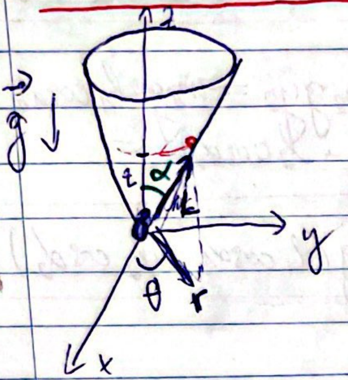
$s$ = # of constraints

$\Rightarrow$ # coordinates $= 3n - s$

$$\Rightarrow \boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = 0}$$

# Point mass in a cone subject to gravity

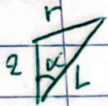
- 3 ~~dimensional~~ dimensional space
  but 1 constraint $\Rightarrow$ 2 degrees of freedom




cylindrical coordinates

$x = r\cos\theta$
$y = r\sin\theta$
$z = z = \frac{r}{\tan\alpha} = r\cot\alpha$

2 coordinates: $r, \theta$




$L\sin\alpha = r$, $L\cos\alpha = z$ $\Big\}$ $\frac{r}{z} = \tan\alpha$

constraint $\Rightarrow g(r,\theta) = r\cot\alpha - z = 0$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2) = \frac{1}{2}m\left[(\dot r\cos\theta - r\sin\theta\,\dot\theta)^2 + (\dot r\sin\theta + r\cos\theta\,\dot\theta)^2 + \dot r^2\cot^2\alpha\right] =$$
$$= \frac{m}{2}\left[\dot r^2\cos^2\theta - \cancel{2r\dot r\cos\theta\sin\theta\,\dot\theta} + r^2\sin^2\theta\,\dot\theta^2 + \dot r^2\sin^2\theta + \cancel{2r\dot r\sin\theta\cos\theta\,\dot\theta} + r^2\cos^2\theta\,\dot\theta^2 + \dot r^2\cot^2\alpha\right]$$
$$= \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2 + \dot r^2\cot^2\alpha\right)$$

$$U = mgz = mgr\cot\alpha$$

$$\Rightarrow L = T - U = \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2 + \dot r^2\cot^2\alpha\right) - mgr\cot\alpha$$

$$\underbrace{\frac{\partial L}{\partial\theta}}_{0} - \frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = 0 \Rightarrow \frac{\partial L}{\partial\dot\theta} = \text{const.} \Rightarrow \boxed{\underbrace{mr^2\dot\theta}_{\text{conserved angular momentum}} = \text{const.}}$$

$\Rightarrow$ $\boxed{\text{conservation of angular momentum}}$

angular momentum about z-axis: $|\vec r\times\vec p|_z = r\cdot m\dot\theta r = r^2 m\dot\theta$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial\dot r} = 0$$

$\dot\theta = \frac{\text{cost}}{mr^2}$

$mgr\dot\theta^2 - mg\cot\alpha$

$$\frac{d}{dt}(m\dot r + m\dot r\cot^2\alpha) = m\ddot r + m\ddot r\cot^2\alpha$$
$$= m\ddot r\left(1 + \frac{\cos^2\alpha}{\sin^2\alpha}\right) = m\ddot r\left(\frac{\sin^2\alpha + \cos^2\alpha}{\sin^2\alpha}\right) = \frac{m\ddot r}{\sin^2\alpha}$$

$$\Rightarrow mgr\dot\theta^2 - mg\cot\alpha - \frac{m\ddot r}{\sin^2\alpha} = 0$$

$$\ddot r - gr\dot\theta^2\sin^2\alpha + g\frac{\cos\alpha}{\sin\alpha}\sin^2\alpha = 0$$

$$\ddot r - gr\dot\theta^2\sin^2\alpha + g\frac{\sin(2\alpha)}{2} = 0$$

$$\ddot r - gr\frac{c^2}{m^2r^4}\sin^2\alpha + g\frac{\sin(2\alpha)}{2} = 0$$

$$\boxed{\ddot r - \frac{gc^2}{m^2r^3}\sin^2\alpha + \frac{g\sin(2\alpha)}{2} = 0}$$
$$\boxed{mr^2\dot\theta = c}$$

equations of motion

# Constraints

- **holonomic constraints** : $g(x_{\alpha,i}; t) = 0$
  - only depends on $x$ (position), not velocities
  - $\alpha$: labelling point mass $\alpha = 1, ..., n$; $i$: labeling spatial indexes

- **non-holonomic** : $g(x_{\alpha,i}, \dot{x}_{\alpha,i}; t) = 0$
  - often can be written as holonomic

$$\sum_i A_i \dot{x}_i + B = 0 \Rightarrow \sum_i \left( \frac{\partial f}{\partial x_i} \dot{x}_i \right) + \frac{\partial f}{\partial t} = 0$$

$$\begin{cases} A_i = \frac{\partial f}{\partial x_i} \\ B = \frac{\partial f}{\partial t} \end{cases} \Rightarrow \sum_i \left( \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} \right) + \frac{\partial f}{\partial t} = 0$$

$$\Rightarrow \frac{df}{dt} = 0$$

$$\Rightarrow f = \text{const. in time}$$

integrate to get holonomic form

$$\Rightarrow g = f - \text{const.} = 0$$

## Lagrange multipliers – multiple constraints

$$g_k(q_{\alpha j}, t) = 0 \quad ; \quad k = 1, ..., s \quad ; \quad j = 1, ..., n$$

for each constraint : lagrange multiplier $\lambda_k$

$$\Rightarrow \boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \sum_{k=1}^{s} \lambda_k(t) \frac{\partial g_k}{\partial q_j} = 0}$$

$\underline{n}$ equations $\underline{+s}$ equations for constraint $\Rightarrow$ and $n+s$ unknowns ($\underline{n}$ $q_j$'s and $\underline{s}$ $\lambda_k$'s)

## Generalised force – usefullness of lagrange multipliers

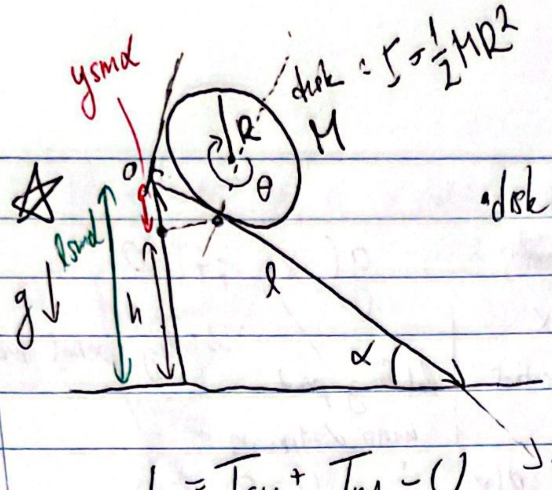
$$\boxed{Q_j \equiv \sum_{k=1}^{s} \lambda_k \frac{\partial g_k}{\partial q_j}} \quad \rightarrow \quad \left[ \frac{E}{\text{length}} \right]$$

- if $q$ = cartesian position coordinate $[L] \Rightarrow [Q] =$ Force

forces **responsible** for constraints

e.g. friction, tension but can also be torque (if $q = \theta$)

↳ **Lagrange multiplier methods is useful, especially when asked about the forces (otherwise that information is lost.**




disk rolling on an incline (with friction)
↳ additional energy from rotation
↳ KE = translational (of CM) + rotational (around central axis)

$$L = T_{CM} + T_{rot} - U$$

$T_{CM} = \frac{1}{2}M\dot{y}^2$, $T_{rot} = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{4}MR^2\dot{\theta}^2$, $U = Mgh = Mg(l-y)\sin\alpha$

Equation of constraint: $y - R\theta = 0$
↳ in general for infinitesimally small $ds = R\,d\theta$ for any shape but $ds$ will vary, e.g. if shape $= x^2 \Rightarrow ds = 2x\,dx$

$$L = \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mg(y-l)\sin\alpha$$

**I** $y = R\theta$

$$= \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mg(R\theta - l)\sin\alpha$$
$$= \frac{3}{4}MR^2\dot{\theta}^2 + Mg(R\theta - l)\sin\alpha$$

$$\underbrace{\frac{\partial L}{\partial \theta}}_{MgR\sin\alpha} - \underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}}_{\frac{3}{2}MR^2\ddot{\theta}} = 0 \Rightarrow MgR\sin\alpha - \frac{3}{2}MR^2\ddot{\theta} = 0$$
$$\Rightarrow \boxed{\ddot{\theta} = \frac{2}{3}\frac{g\sin\alpha}{R}}$$
$$\Rightarrow \theta = 3\frac{g\sin\alpha}{R}t^2 + At + B$$

**II** using Lagrange multipliers

$$\begin{cases} \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial g}{\partial \theta} = 0 \Rightarrow \boxed{-\frac{1}{2}MR^2\ddot{\theta} - \lambda R = 0} \; ① \\ \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda\frac{\partial g}{\partial y} = 0 \Rightarrow \boxed{Mg\sin\alpha - M\ddot{y} + \lambda = 0} \; ② \\ g = \boxed{y - R\theta = 0} \; ③ \Rightarrow \ddot{y} = R\ddot{\theta} \end{cases}$$

$$-\frac{1}{2}MR\ddot{\theta} + Mg\sin\alpha - MR\ddot{\theta} = 0$$
$$\frac{3}{2}MR\ddot{\theta} = Mg\sin\alpha$$

and

$$\boxed{\lambda = -\frac{Mg}{3}\sin\alpha} \quad \boxed{\ddot{y} = \frac{2}{3}g\sin\alpha} \quad \boxed{\ddot{\theta} = \frac{2}{3}\frac{g\sin\alpha}{R}} \xleftarrow{\ddot{y} = R\ddot{\theta}}$$

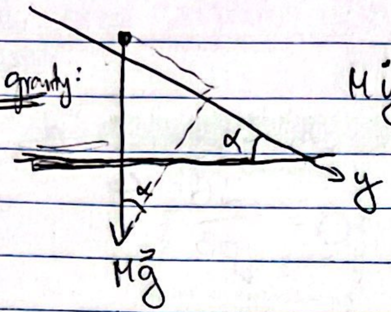
$$Q_j = \sum_{k=1}^{s}\lambda_k\frac{\partial g_k}{\partial q_j}$$

cartesian coordinate ⇒ actual force

$$\begin{cases} Q_y = \lambda\frac{\partial g}{\partial y} = -\frac{Mg}{3}\sin\alpha \rightsquigarrow \text{friction force} : f = \frac{1}{3} \\ Q_\theta = \lambda\frac{\partial g}{\partial \theta} = +\frac{Mg}{3}\sin\alpha\, R \rightsquigarrow \text{torque exerted by friction} \end{cases}$$

generalised forces

Only gravity:




$M\ddot{y} = Mg\sin\alpha$

+Friction: $\ddot{y}M = \frac{2}{3}Mg\sin\alpha$

Newton: $\ddot{y}M = F_G + F_f = Mg\sin\alpha + F_f = \frac{2}{3}Mg\sin\alpha$

$\Rightarrow F_f = -\frac{1}{3}Mg\sin\alpha$

$\Rightarrow f = \frac{1}{3}$ → independent of material ?! ~~...~~ not really, this is for the case of rolling

$\Rightarrow$ only if $f \geq 1/3$

Use Lagrange multipliers only if asked to find the generalized forces

# Equivalence of Newtonian with Lagrangian Approach

Newtonian → Forces → vectors

Lagrangian → Energy (Lagrangian) → scalar

## Lagrangian ⇒ Newtonian

- use cartesian coordinates: $q_i = x_i$ , $i = 1,2,3$

$$\begin{cases} T = T(\dot{x}_i) \\ U = U(x_i) \end{cases}$$

indep. $\Rightarrow \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$

$$\Rightarrow \frac{\partial (T-U)}{\partial x_i} - \frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{x}_i} = 0$$

$$-\frac{\partial U}{\partial x_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = 0$$

and $F_i = -\frac{\partial U}{\partial x_i}$ $\Rightarrow$ $F_i = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i}$

and $T = \sum_j \frac{1}{2} m \dot{x}_j^2$

$\frac{\partial T}{\partial \dot{x}_i} = m\dot{x}_i$ → $F_i = \frac{d}{dt} m\dot{x}_i$

$\Rightarrow \boxed{F_i = m\ddot{x}_i} = \dot{p}_i$ □

# Newtonian ⇒ Lagrangian

$x_i = x_i(q_j, t)$

- $\frac{dx_i}{dt} = \dot{x}_i = \sum_k \left(\frac{\partial x_i}{\partial q_k}\dot{q}_k\right) + \frac{\partial x_i}{\partial t}$

- $\frac{\partial}{\partial \dot{q}_j}(\dot{x}_i) = \frac{\partial x_i}{\partial q_j}$

- generalised momentum $\boxed{p_j \equiv \frac{\partial T}{\partial \dot{q}_j} \overset{\text{if } U \text{ indep. of } \dot{q}}{=} \frac{\partial L}{\partial \dot{q}_j}}$

- work $= \delta W = \sum_i F_i \, \delta x_i = \sum_i F_i \sum_j \frac{\partial x_i}{\partial q_j}\delta q_j =$

spatial displacement

$= \sum_j \boxed{\left(\sum_i F_i \frac{\partial x_i}{\partial q_j}\right)} \delta q_j$

$\boxed{Q_j = -\frac{\partial U}{\partial q_j}}$ ⟵ "$Q_j$" = generalised force wrt $q_j$ ⟶ $Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} = \sum_i \dot{p}_i \frac{\partial x_i}{\partial q_j} = \sum_i m \ddot{x}_i \frac{\partial x_i}{\partial q_j}$

- $p_j \equiv \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left[\frac{1}{2} m \sum_i \dot{x}_i^2\right] =$

$= \frac{m}{2}\sum_i 2\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} = m \sum_i \dot{x}_i \frac{\partial x_i}{\partial q_j} =$

- $\frac{d}{dt} p_j = \frac{d}{dt} m \sum_i \dot{x}_i \frac{\partial x_i}{\partial q_j} = m \sum_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}\right) =$

$= m \sum_i \left[\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \dot{x}_i \left(\frac{\partial^2 x_i}{\partial q_j \partial t} + \sum_k \frac{\partial^2 x_i}{\partial q_j \partial q_k}\dot{q}_k\right)\right]$

- $\frac{\partial T}{\partial q_j} = \sum_i m \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j} = \sum_i m \dot{x}_i \frac{\partial}{\partial q_j}\left[\sum_k \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}\right] =$

$= \sum_i m \dot{x}_i \left(\frac{\partial^2 x_i}{\partial t \partial q_j} + \sum_k \dot{q}_k \frac{\partial^2 x_i}{\partial q_j \partial q_k}\right)$

$\Rightarrow \frac{d}{dt} p_j = \sum_i m \ddot{x}_i \frac{\partial x_i}{\partial q_j} + \frac{\partial T}{\partial q_j} = Q_j + \frac{\partial T}{\partial q_j}$

$\Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = Q_j + \frac{\partial T}{\partial q_j} = \frac{\partial T}{\partial q_j} - \frac{\partial U}{\partial q_j} = \frac{\partial (T-U)}{\partial q_j} = \frac{\partial L}{\partial q_j}$

$\frac{\partial U}{\partial \dot{q}_j} = 0$, so we can subtract it freely $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial U}{\partial \dot{q}_j}\right) = \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}}$ □

# Conservation laws

① if $L$ indep. of one of the coordinates, i.e.: $L = L(q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$

$\Rightarrow \boxed{\frac{\partial L}{\partial q_j} = 0}$ $\Rightarrow \underbrace{\frac{\partial L}{\partial q_j}}_{0} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$

$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \Rightarrow \boxed{\frac{\partial L}{\partial \dot{q}_j} = \text{const.}}$

$\Rightarrow$ conservation of gen. momentum

conserved quantity

generalised momentum $\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j}}$

☆ for $q_j = x_j \Rightarrow$ momentum conservation if no dependence on $x_j$

☆ for $q_j = \theta \Rightarrow$ angular momentum conservation if no dependence on $\theta$.

② if $L$ independent of time $\Rightarrow$ isolated system $\Rightarrow \boxed{\frac{\partial L}{\partial t} = 0}$

invariant under time translation

$\rightsquigarrow$ ~~[illegible]~~ conserved quantity:

Hamiltonian $\boxed{H \equiv \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L}$

doesn't depend explicitly on time

$$\frac{d}{dt} L = \underbrace{\frac{\partial L}{\partial t}}_{0} + \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

$\boxed{H = \sum_j \dot{q}_j p_j - L}$

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial \frac{dL}{dt}}{\partial \frac{dq_j}{dt}}$$

$$\frac{d}{dt} L = \sum_j \left( \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \right) = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j$$

implicit summation

$$\Rightarrow \frac{d}{dt}\Big(\underbrace{L - \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j}_{= -H}\Big) = 0 \quad \Rightarrow H \text{ is conserved in time}$$

$\rightsquigarrow$ interpretation: $H = E = T + U$ but only under following conditions

① $\frac{\partial U}{\partial \dot{q}_j} = 0$ : potential energy is indep. of generalised velocity

② $x_j = x_j(q_j, \cancel{t})$ : coordinate transformation is independent of explicit time

$\Rightarrow$ ~~then $H$ is conserved~~

$H = E = T + U$

## Prove $H = E = T + U$ :

#spatial dimen

• $T = \frac{1}{2}\sum_{\alpha=1}^{n}\left(m_\alpha \sum_{i=1}^{3} \dot{x}_{\alpha i}^2\right)$ , $x_{\alpha i} = x_{\alpha i}(q_j, t)$

n=#point masses

$$\left(\frac{dx_{\alpha i}}{dt}\right)^2 = \dot{x}_{\alpha i}^2 = \left(\frac{\partial x_{\alpha i}}{\partial t} + \sum_j \frac{\partial x_{\alpha i}}{\partial q_j}\underbrace{\frac{dq_j}{dt}}_{=\dot{q}_j}\right)^2 =$$

$$= \left(\frac{\partial x_{\alpha i}}{\partial t}\right)^2 + \sum_{j,k}\frac{\partial x_{\alpha i}}{\partial q_j}\dot{q}_j \frac{\partial x_{\alpha i}}{\partial q_k}\dot{q}_k$$

$$+ 2\frac{\partial x_{\alpha i}}{\partial t}\sum_j \frac{\partial x_{\alpha i}}{\partial q_j}\dot{q}_j - b_j$$

$= a_{jk}$

$$T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \sum_{i=1}^{3}\left[\underbrace{\left(\frac{\partial x_{\alpha i}}{\partial t}\right)^2}_{c} + 2\frac{\partial x_{\alpha i}}{\partial t}\sum_j \frac{\partial x_{\alpha i}}{\partial q_j}\dot{q}_j + \sum_{j,k}\frac{\partial x_{\alpha i}}{\partial q_j}\dot{q}_j\frac{\partial x_{\alpha i}}{\partial q_k}\dot{q}_k\right]$$

T as polynomial in gen velocities.

$$= \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha\left(\sum_{j,k}\dot{q}_j\dot{q}_k a_{jk} + \sum_j \dot{q}_j b_j + c\right)$$

$a_{jk} = \frac{\partial x_{\alpha i}}{\partial q_j}\frac{\partial x_{\alpha i}}{\partial q_k}$ , $b_j = 2\frac{\partial x_{\alpha i}}{\partial t}\frac{\partial x_{\alpha i}}{\partial q_j}$ , $c = \left(\frac{\partial x_{\alpha i}}{\partial t}\right)^2$

use assumption that $\frac{\partial x_i}{\partial t} = 0$ (2) $\Rightarrow$ $b_j = 0$ , $c = 0$

$$\Rightarrow T = \frac{1}{2}\sum_{\alpha=1}^{n} m_\alpha \sum_{j,k}\dot{q}_j\dot{q}_k a_{jk}$$

independent coordinates, kronecker delta

• $\frac{\partial T}{\partial \dot{q}_i} = \sum_{j,k} a_{jk}\left[\dot{q}_k\frac{\partial \dot{q}_j}{\partial \dot{q}_i} + \dot{q}_j\frac{\partial \dot{q}_k}{\partial \dot{q}_i}\right] = \sum_k\sum_j a_{jk}\left[\dot{q}_k\delta_{ji} + \dot{q}_j\delta_{ki}\right] =$

$= \sum_k a_{ik}\dot{q}_k + \sum_j a_{ji}\dot{q}_j \underset{\uparrow}{=} 2\sum_k a_{ik}\dot{q}_k$

$a_{ik} = a_{ji}$ (symmetric)

$$\sum_i \dot{q}_i\frac{\partial T}{\partial \dot{q}_i} = 2\underbrace{\sum_{i,k}\dot{q}_i\dot{q}_k a_{ik}\frac{m_\alpha}{2}}_{T}$$

$$\Rightarrow \boxed{2T = \sum_i \dot{q}_i\frac{\partial T}{\partial \dot{q}_i}} \rightsquigarrow \frac{\partial T}{\partial \dot{q}_i} = \frac{2T}{\dot{q}_i}$$

using assumption (1)

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}(T-U) = \frac{\partial T}{\partial \dot{q}_j} = \frac{2T}{\dot{q}_j}$$

$$\Rightarrow \underline{H} \equiv \dot{q}_j\frac{\partial L}{\partial \dot{q}_j} - L = \dot{q}_j\frac{2T}{\dot{q}_j} - L = 2T - L = 2T - (T-U) =$$

$$= 2T - T + U = \underline{T+U}$$

□

$$\boxed{\frac{\partial H}{\partial q_j} = -\dot{p}_j \,, \quad \frac{\partial H}{\partial p_j} = \dot{q}_j}$$

# Hamilton's equations

8/12/2022

↳ equivalent to Lagrange's equations

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \quad , \quad \text{gen. momentum: } \boxed{p_j := \frac{\partial L}{\partial \dot{q}_j}} \to \dot{q}(p, q)$$

n 2nd order equations in time

$$H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \quad \Leftarrow \quad H = \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$\boxed{H = \dot{q}_j p_j - L(q_j, \dot{q}_j, t)}$$

when index is twice in the same term (multiplied) ⇒ implicit summation

## Lagrange → Hamilton's

$H = \dot{q}_j p_j - L(q_j, \dot{q}_j, t)$ but $\dot{q}_j$ and $p_j$ are related by $p_j = \frac{\partial L}{\partial \dot{q}_j}$

so we can rewrite $H$ as function of:

$$\Rightarrow \quad H = H(q_j, p_j, t) \text{ with } j = 1, \dots, n$$

$$\Rightarrow \quad dH = \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt$$

and

$$dH = \sum \dot{q}_j\, dp_j + \cancel{\sum p_j\, d\dot{q}_j} - \left( \sum \boxed{\frac{\partial L}{\partial q_j}} dq_j + \cancel{\sum \boxed{\frac{\partial L}{\partial \dot{q}_j}} d\dot{q}_j} + \frac{\partial L}{\partial t} dt \right)$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial q_j} = \frac{d}{dt} p_j = \dot{p}_j$$

$$= \dot{q}_j\, dp_j - \dot{p}_j\, dq_j - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \quad \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt = \dot{q}_j\, dp_j - \dot{p}_j\, dq_j - \frac{\partial L}{\partial t} dt$$

↳ ~~to be~~ equal ⇔ each $dx_i$ term equal

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{\partial H}{\partial q_j} &= -\dot{p}_j \\ \frac{\partial H}{\partial p_j} &= \dot{q}_j \end{aligned}}$$

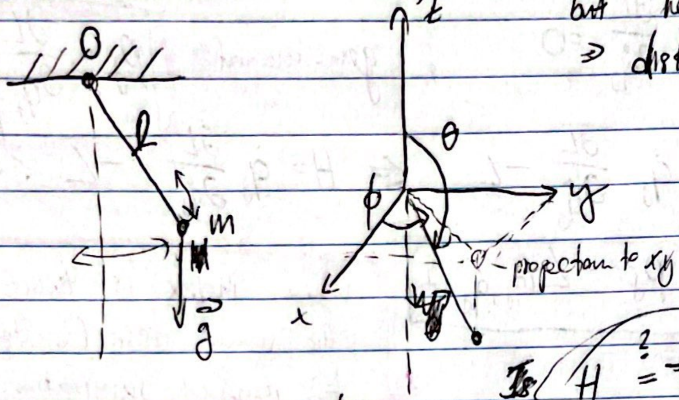
Hamilton's equations

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

⇝ more on p. 30

## Spherical pendulum

- can move in 3D but restriction by $L$ $\Rightarrow$ distance from $O$ $r = l$ always $\Rightarrow$ 2 indep. coordinates: $\theta, \phi$




Is $H \overset{?}{=} T+U$ (2)

- forces: tension $T$ (already used as constraint)
  gravity $\Rightarrow U = +mgl\cos\theta$ indep. of $\dot\phi, \dot\theta$ $\Rightarrow$ (1) ✓
- coordinate transforms indep. of $t \Rightarrow$ (2) ✓

conditions satisfied $\Rightarrow \boxed{H = T+U}$

$$\begin{cases} x = l\sin\theta\cos\phi \\ y = l\sin\theta\sin\phi \\ z = l\cos\theta \end{cases}$$

$$\begin{cases} \dot x = l\cos\theta\,\dot\theta\cos\phi + l\sin\theta\,\dot\phi(-\sin\phi) \\ \dot y = l\cos\theta\,\dot\theta\sin\phi + l\sin\theta\cos\phi\,\dot\phi \\ \dot z = -l\sin\theta\,\dot\theta \end{cases}$$

$$T = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2) = \frac{1}{2}m\big(l^2\cos^2\theta\cos^2\phi\,\dot\theta^2 - \cancel{2l^2\dot\theta\dot\phi\cos\theta\sin\theta\cos\phi\sin\phi}$$
$$+ l^2\sin^2\theta\,\dot\phi^2\sin^2\phi + l^2\cos^2\theta\,\dot\theta^2\sin^2\phi + \cancel{2l^2\dot\theta\dot\phi\sin\theta\cos\theta\sin\phi\cos\phi}$$
$$+ l^2\sin^2\theta\cos^2\phi\,\dot\phi^2 + l^2\sin^2\theta\,\dot\theta^2\big) =$$
$$= \frac{1}{2}m\left(l^2\cos^2\theta\,\dot\theta^2 + l^2\sin^2\theta\,\dot\phi^2 + l^2\sin^2\theta\,\dot\theta^2\right) =$$
$$= \frac{1}{2}m\left(l^2\dot\theta^2 + l^2\sin^2\theta\,\dot\phi^2\right)$$

$U = mgl\cos\theta$ ($\hookrightarrow 0$ at top) or $U = mgl(1+\cos\theta)$ ($\rightsquigarrow 0$ at bottom)

$$\Rightarrow H = T+U = \frac{1}{2}m\left(l^2\dot\theta^2 + l^2\sin^2\theta\,\dot\phi^2\right) + mgl\cos\theta \;\}$$

but now $H = H(\theta, \dot\theta, \dot\phi)$ but we want $H = H(\theta, \phi, p_\theta, p_\phi)$

$$L = \frac{1}{2}m\left(l^2\dot\theta^2 + l^2\sin^2\theta\,\dot\phi^2\right) - mgl\cos\theta \;\}$$

**Find $p_\theta, p_\phi$:**

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = ml^2\dot\theta \Rightarrow \dot\theta = \frac{p_\theta}{ml^2}$$
$$p_\phi = \frac{\partial L}{\partial\dot\phi} = ml^2\sin^2\theta\,\dot\phi \Rightarrow \dot\phi = \frac{p_\phi}{ml^2\sin^2\theta}$$

$$\Rightarrow H = \frac{1}{2}m\left(l^2\frac{p_\theta^2}{m^2l^4} + l^2\sin^2\theta\frac{p_\phi^2}{m^2l^4\sin^4\theta}\right) + mgl\cos\theta \;\}$$
$$= \frac{1}{2}\left(\frac{p_\theta^2}{ml^2} + \frac{p_\phi^2}{ml^2\sin^2\theta}\right) + mgl\cos\theta \;\}$$
$$= \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta} + mgl\cos\theta$$

We have $H$, so we apply Hamilton's equations:

$$\begin{cases} \dfrac{\partial H}{\partial q_j} = -\dot{p}_j \\[2mm] \dfrac{\partial H}{\partial p_j} = \dot{q}_j \end{cases} \quad \text{and} \quad H = \frac{p_\theta^2}{2ml^2} + \frac{p_\phi^2}{2ml^2\sin^2\theta} + mgl\cos\theta$$

$$-\frac{p_\phi^2 \, 2ml^2 \, 2\cos\theta\sin\theta}{(2ml^2\sin^2\theta)^2}$$

$$\Rightarrow \begin{cases} \dfrac{\partial H}{\partial \theta} = -mgl\sin\theta \checkmark \equiv -\dot{p}_\theta = -ml^2\ddot{\theta} \\[2mm] \dfrac{\partial H}{\partial \phi} = 0 \equiv -\dot{p}_\phi = -ml^2 2\sin\theta\cos\theta\dot{\theta}\dot{\phi} - ml^2\sin^2\theta\ddot{\phi} \\[2mm] \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{ml^2} = \dfrac{ml^2\dot{\theta}}{ml^2} = \dot{\theta} \equiv \dot{\theta} \\[2mm] \dfrac{\partial H}{\partial p_\phi} = \dfrac{p_\phi}{ml^2\sin^2\theta} = \dfrac{ml^2\sin^2\theta\dot{\phi}}{ml^2\sin^2\theta} = \dot{\phi} \equiv \dot{\phi} \end{cases}$$

$$\frac{\partial H}{\partial \theta} : -mgl\sin\theta - \frac{m^2 l^4 \sin^4\theta\dot{\phi}^2 \, 2ml^2 \, 2\cos\theta\sin\theta}{4m^2l^4\sin^4\theta} =$$

$$= -mgl\sin\theta - ml^2\cos\theta\sin\theta\dot{\phi}^2 = -ml^2\ddot{\theta}$$

$$\Rightarrow \frac{g}{l}\sin\theta + \cos\theta\sin\theta\dot{\phi}^2 = \ddot{\theta}$$ as from Lagrange

$$\frac{\partial H}{\partial \phi} : 0 = 2\cos\theta\dot{\theta}\dot{\phi} + \sin\theta\ddot{\phi}$$

$$\begin{cases} \dot{p}_\theta = -mgl\sin\theta - \dfrac{p_\phi^2 \, 2\cos\theta}{2ml^2\sin^3\theta} \\[2mm] \dot{p}_\phi = 0 \Rightarrow \text{const. } p_\phi \Rightarrow \boxed{\text{conservation of angular momentum about z-axis}} \\[2mm] \dfrac{p_\theta}{ml^2} = \dot{\theta} \\[2mm] \dfrac{p_\phi}{ml^2\sin^2\theta} = \dot{\phi} \end{cases}$$

Hamiltons equations of motion

$$\boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

I $\therefore dH = \frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt$

$-\dot{p}_j$ ; $\dot{q}_j \rightarrow \frac{\partial H}{\partial p_j} = \frac{dq_j}{dt} \Rightarrow \boxed{dq_j = \frac{\partial H}{\partial p_j} dt}$

$$\frac{\partial H}{\partial q_j} = -\frac{dp_j}{dt} \Rightarrow \boxed{dp_j = -\frac{\partial H}{\partial q_j} dt}$$

$$\Rightarrow dH = \underbrace{+\frac{\partial H}{\partial q_j}\frac{\partial H}{\partial p_j} dt - \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial q_j} dt}_{\Rightarrow 0} + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow dH = \frac{\partial H}{\partial t} dt \Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$

$\square$

proof that if $L$ is independent of time explicitly, ~~then~~ $\therefore$ ($\Rightarrow \frac{\partial L}{\partial t} = 0$), then $\frac{dH}{dt} = 0$

$\Rightarrow$ $H$ is conserved in time.

II $\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y' \frac{\partial f}{\partial y'}\right) = 0$ : Euler's eq - 2nd form

for $L$: $\frac{\partial L}{\partial t} - \frac{d}{dt}\left(L - \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$

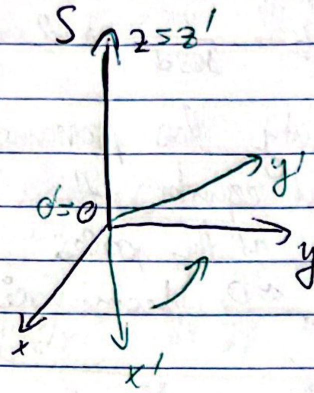
$$\Rightarrow \frac{\partial L}{\partial t} = \frac{d}{dt}\left(L - \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = -\frac{dH}{dt}$$

$\square$

$\Rightarrow \boxed{-\frac{\partial L}{\partial t} = \frac{dH}{dt}}$ alternative way of obtaining the same result

# Non-inertial Frames

- for inertial frames: $\vec{F}_{real} = m\vec{a}$
- for non-inertial frames: $m\vec{a}' = \vec{F}_{real} + \vec{F}_{fictitious}$




S = inertial
S' = rotating (non-inertial)

$\vec{\omega} = \omega\hat{z}$ - angular velocity for S'

* $\boxed{\dot{\vec{r}} = \vec{\omega} \times \vec{r}}$ holds for any position vector $\vec{r}$ *

rotational velocity of a point mass in S' as seen by observer from S

* in particular, also for unit basis vectors $\hat{e}_1'$, $\hat{e}_2'$, $\hat{e}_3'$

$$\left.\begin{aligned}\dot{\hat{e}}_1' &= \vec{\omega} \times \hat{e}_1' \\ \dot{\hat{e}}_2' &= \vec{\omega} \times \hat{e}_2' \\ \dot{\hat{e}}_3' &= \vec{\omega} \times \hat{e}_3'\end{aligned}\right\}$$

➤ $\vec{a} = \ddot{\vec{r}}$ - acceleration in inertial frame ⓘ relate $\vec{a}$ and $\vec{a}'$

➤ $\vec{r} = \sum_i r_i \hat{e}_i = r_i \hat{e}_i \quad \rightarrow \quad \dot{\vec{r}} = \dot{r}_i \hat{e}_i$

$\vec{r} = \sum_i r_i' \hat{e}_i' = r_i' \hat{e}_i' \quad \rightarrow \quad \dot{\vec{r}} = \dot{r}_i' \hat{e}_i' + r_i' \dot{\hat{e}}_i' = \dot{r}_i' \hat{e}_i' + r_i' \underbrace{\vec{\omega} \times \hat{e}_i'}_{\vec{\omega} \times (r_i' \hat{e}_i') = \vec{\omega} \times \vec{r}}$

same vector, written wrt two different bases

➤ $\ddot{\vec{r}} = \ddot{r}_i' \hat{e}_i' + \dot{r}_i' \underset{\vec{\omega} \times \hat{e}_i'}{\underset{\shortparallel}{\dot{\hat{e}}_i'}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \underset{\dot{r}_i' \hat{e}_i' + \vec{\omega} \times \vec{r}}{\underset{\shortparallel}{\dot{\vec{r}}}}$

$= \ddot{r}_i' \hat{e}_i' + \dot{r}_i' (\vec{\omega} \times \hat{e}_i') + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{r}_i' \hat{e}_i' + \vec{\omega} \times (\vec{\omega} \times \vec{r})$

$$\left.\frac{d^2\vec{r}}{dt^2}\right|_{(S)} = \left.\frac{d^2\vec{r}}{dt^2}\right|_{(S')} + \underbrace{2\dot{r}_i'\, \vec{\omega} \times \hat{e}_i'}_{= 2\vec{\omega} \times (\dot{r}_i' \hat{e}_i') = 2\vec{\omega} \times \underbrace{\left.\frac{d\vec{r}}{dt}\right|_{(S')}}_{= \vec{v}'}} + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$a$ ‖ ; acceleration as seen from S' frame $= a'$

$\vec{r}$ for rotating ‖

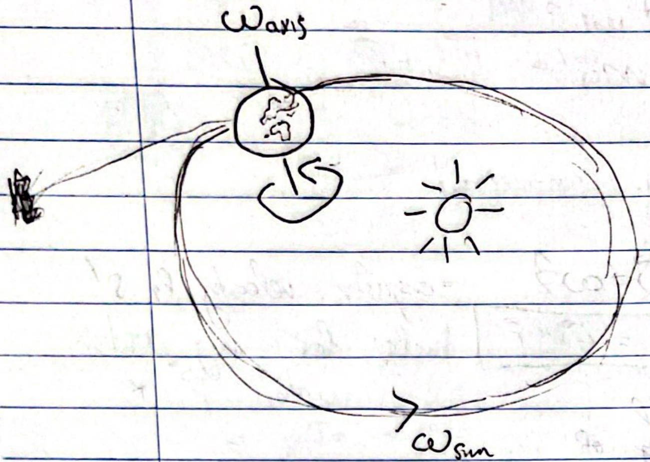
$$\Rightarrow \vec{a} = \vec{a}' + 2\vec{\omega} \times \vec{v}' + \dot{\vec{\omega}} \times \vec{r}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}')$$

$\vec{F}_{real} = m\vec{a}$, so

$$\boxed{m\vec{a}' = \vec{F}_{real} \underbrace{\overbrace{- 2(\vec{\omega} \times \vec{v}')m}^{\text{coriolis force}} \overbrace{- (\dot{\vec{\omega}} \times \vec{r}')m}^{\text{euler force}} \overbrace{- \vec{\omega} \times (\vec{\omega} \times \vec{r}')\, m}^{\text{centrifugal / centripetal force}} + m\ddot{\vec{R}}}_{\text{Fictitious forces}}} = \vec{F}_{real} + \vec{F}_{fictitious}$$

if whole frame accelerating with $\ddot{\vec{R}}$

## ✡ Earth as a non-inertial frame

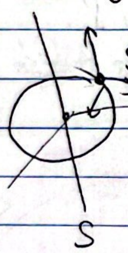



$\omega_{axis} = \frac{2\pi}{24h} = 7 \times 10^{-5}\ s^{-1}$

$\omega_{sun} = \frac{2\pi}{365d} = \cancel{2 \times 10^{-7}\ s^{-1}}$ small

- gravity less pronounced at equator than at the poles
- $F_{Euler} \approx 0$ because $\dot{\omega}$ small

centrifugal force:




$m a' = \vec{F} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}') - 2m\vec{\omega} \times (\vec{\omega} \times \vec{v}') - \ddot{\vec{R}} m$

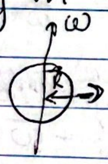
$\dot{\vec{R}} = \vec{\omega} \times \vec{R}$

$\ddot{\vec{R}} = \vec{\omega} \times \dot{\vec{R}}$

$= \vec{F} - m\vec{\omega} \times \cancel{(\vec{\omega} \times \vec{r}')}^{0} - 2m \cancel{(\vec{\omega} \times \vec{v}')}^{0}$
$- m\vec{\omega} \times (\vec{\omega} \times \vec{R})$

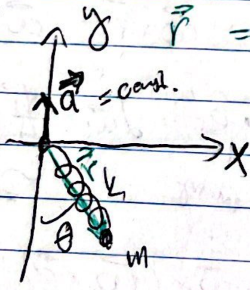
- m close to surface $\Rightarrow |\vec{r}'| << R$
- gravity $= m\vec{g}_0$

$\vec{g}' m \approx m\vec{g}_0 - m\vec{\omega} \times (\vec{\omega} \times \vec{R})$




reducing pull of gravity but 0 at poles
$\Rightarrow \vec{g}$ stronger at poles

## ✡




$\vec{r}$ = distance of mass to fulcrum

Inertial frame

$$\begin{cases} x = r\sin\theta \\ y = -r\cos\theta + \frac{1}{2}at^2 \end{cases}$$

depends explicitly on t

$\Rightarrow H \neq E_{total}$ breaks condition ②

$\dot{x}^2 = |\dot{r}\sin\theta + r\cos\theta\dot{\theta}|^2 = \dot{r}^2\sin^2\theta + \cancel{2}\, r\dot{r}\sin(2\theta)\dot{\theta} + r^2\cos^2\theta\dot{\theta}^2$

$\dot{y}^2 = |-\dot{r}\cos\theta + r\sin\theta\dot{\theta} + at|^2 = \dot{r}^2\cos^2\theta - 2\dot{r}\cos\theta(r\sin\theta\dot{\theta} + at)$
$+ r^2\sin^2\theta\dot{\theta}^2 + a^2t^2 + 2atr\sin\theta\dot{\theta}$

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 - 2\dot{r}\cos\theta at + a^2t^2 + 2atr\sin\theta\dot{\theta})$
$= \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + a^2t^2 + 2at(r\sin\theta\dot{\theta} - \dot{r}\cos\theta)\right]$

$r_0$ = rest length of spring

$$U = U_{spring} + \underset{\overset{\shortparallel}{mgy}}{U_{gravity}} = \frac{1}{2}k(r-r_0)^2 - mgr\cos\theta + \frac{1}{2}mgat^2$$

$$L = T - U = \frac{m}{2}\Big[\dot r^2 + r^2\dot\theta^2 + a^2t^2 + 2atr\dot\theta\sin\theta - 2at\dot r\cos\theta$$
$$- \frac{k}{m}(r-r_0)^2 + 2gr\cos\theta - gat^2\Big)$$

$\frac{\partial L}{\partial t} \neq 0 \Rightarrow \frac{dH}{dt} \neq 0 \Rightarrow H$ is not conserved in time

$$\boxed{\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} = 0}$$

$$r\dot\theta^2 + at\dot\theta\sin\theta + g\cos\theta - \frac{k}{m}(r-r_0) - \ddot r + at\sin\theta\,\dot\theta = 0$$
$+ a\cos\theta$

$$\boxed{\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 0}$$

$$0 = matr\dot\theta\cos\theta + mat\dot r\sin\theta - mgr\sin\theta - \frac{d}{dt}\left[mr^2\dot\theta + matr\sin\theta\right] =$$
$$0 = 2atr\dot\theta\cos\theta + \cancel{at\dot r\sin\theta} - mgr\sin\theta - 2r\dot\theta\dot r - r^2\ddot\theta - ar\sin\theta - \cancel{at\sin\theta\,\dot r} + \cancel{atr\dot\theta\cos\theta}$$

$\boxed{H \neq T + U}$ $\quad H = \dot q p - L = \dot\theta p_\theta + \dot r p_r - L$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = \frac{1}{2}m(2r^2\dot\theta^2 + 2atr\sin\theta) = mr(r\dot\theta^2 + at\sin\theta)$$

$$\Rightarrow \boxed{\frac{p_\theta}{mr^2} - \frac{at\sin\theta}{r} = \dot\theta}$$ replace in $L$

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r - mat\cos\theta \Rightarrow \boxed{\dot r = \frac{p_r}{m} + at\cos\theta}$$

$\Rightarrow$ obtain $H$ $\Rightarrow$ Hamilton's eq.

Non-inertial frame

$$\begin{cases} x' = r\sin\theta & \rightarrow \dot x' = \dot r\sin\theta + r\dot\theta\cos\theta \\ y' = -r\cos\theta & \rightarrow \dot y' = -\dot r\cos\theta + r\dot\theta\sin\theta \end{cases}$$

$$T = (\dot x'^2 + \dot y'^2)\frac{m}{2} = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2)$$

$$U = U_{grav} + U_{spring} = -m(g+a)r\cos\theta + \frac{1}{2}k(r-r_0)^2$$

$\Rightarrow$ in non-inertial frame: conditions ① ✓, ② ✓ satisfied

$\Rightarrow$ $H = E$ and $\frac{dH}{dt} = 0$ (conserved in time)

# Rigid body

= collection of
- point masses within fixed distance of each other
- discrete

$$\sum_{\alpha=1}^{N} m_\alpha = M$$

$\leadsto$ mass of point masses

or consider continuous:

$$M = \int_V dm$$

# of independent degrees of freedom: at most 6

- 3 translation + 3 rotation

**Center of mass**

discrete: $$\boxed{\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha}$$ $\hookrightarrow$ position of point mass $\alpha$

continuous: $$\boxed{\vec{R} = \frac{1}{M}\int_V \vec{r}\, dm}$$

**Total momentum**

$$\vec{p} = \sum_{\alpha=1}^{N} \vec{p}_\alpha = \sum_{\alpha=1}^{N} m_\alpha \vec{v}_\alpha = \sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha =$$

$$= \sum_{\alpha=1}^{N} \frac{d m_\alpha \vec{r}_\alpha}{dt} = \frac{d}{dt}\underbrace{\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha}_{M\vec{R}} = \frac{d}{dt} M\vec{R} = M\dot{\vec{R}}$$

$$\Rightarrow \boxed{\vec{p} = M\dot{\vec{R}}}$$

$$\dot{\vec{p}} = \sum_{\alpha=1}^{N} m_\alpha \frac{d^2 \vec{r}_\alpha}{dt^2} = \sum_{\alpha=1}^{N} \vec{F}_\alpha = \sum_{\alpha=1}^{N}\left(\cancel{\vec{F}_{\alpha,int}} + \vec{F}_{\alpha,ext}\right)$$

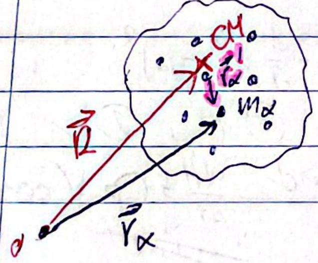
$$= \sum_{\alpha=1}^{N} \vec{F}_{\alpha,ext} =: \vec{F}_{ext}$$

0 due to all other point mass

$\Rightarrow$ from Newton's 3rd law all of these cancel out

$$\Rightarrow \boxed{\dot{\vec{p}} = \sum_{\alpha=1}^{N} \vec{F}_{\alpha,ext} = \vec{F}_{ext}}$$

**Total angular momentum**




$$\boxed{\vec{L} = \underbrace{\vec{R}\times\vec{p}}_{\text{motion of the whole body}} + \underbrace{\sum_\alpha \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha}_{\text{rotation wrt CM}}}$$

$$\boxed{\vec{R} + \underbrace{\vec{r}_\alpha^{\,\prime}}_{\text{wrt CM}} = \vec{r}_\alpha} \qquad \boxed{\sum_{\alpha=1}^{N} \vec{r}_\alpha^{\,\prime} m_\alpha = 0}$$

$$\sum_{\alpha=1}^{N} m_\alpha(\vec{r}_\alpha - \vec{R}) = \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha - \vec{R}\sum_{\alpha=1}^{N} m_\alpha = \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha - \vec{R}M = M\vec{R} - M\vec{R} = 0$$

$$\vec{L} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha = \sum_{\alpha=1}^{N} m_\alpha (\vec{R} + \vec{r}_\alpha^{\,\prime}) \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha^{\,\prime}) =$$

$$= \sum_{\alpha=1}^{N} m_\alpha (\vec{R} \times \dot{\vec{R}} + \vec{r}_\alpha^{\,\prime} \times \dot{\vec{r}}_\alpha^{\,\prime} + \vec{R} \times \dot{\vec{r}}_\alpha^{\,\prime} + \vec{r}_\alpha^{\,\prime} \times \dot{\vec{R}})$$

$$= \vec{R} \times (M\dot{\vec{R}}) + \sum m_\alpha \vec{r}_\alpha^{\,\prime} \times \dot{\vec{r}}_\alpha^{\,\prime}$$

$$\sum m_\alpha$$

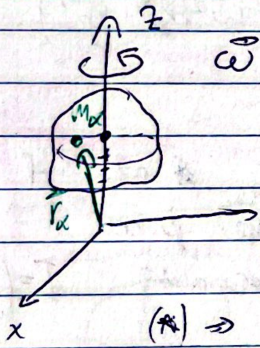
$$\vec{R} \times \sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha^{\,\prime} = \vec{R} \times \frac{d}{dt}\left(\sum m_\alpha \vec{r}_\alpha^{\,\prime}\right) = 0 \qquad -\dot{\vec{R}} \times \underbrace{\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha^{\,\prime}}_{=0} \Rightarrow 0$$

$$\Rightarrow \vec{L} = M\,\vec{R} \times \dot{\vec{R}} + \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha^{\,\prime} \times \dot{\vec{r}}_\alpha^{\,\prime}$$

## Kinetic energy

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha^{\,\prime 2}$$

## Moment of Inertia




$$\vec{\omega} = (0, 0, \omega)$$

$$\vec{L} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times m\vec{v}_\alpha \quad (*)$$

only rotation: $\vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha$

$$(*) \Rightarrow \vec{L} = \sum_{\alpha=1} m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = (-\omega y_\alpha, \omega x_\alpha, 0)$$

distance² to axis of rotation

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_\alpha & y_\alpha & z_\alpha \\ -\omega y_\alpha & \omega x_\alpha & 0 \end{vmatrix} = (-\omega x_\alpha z_\alpha, -\omega y_\alpha z_\alpha, \omega(x_\alpha^2 + y_\alpha^2))$$

moment of inertia

$$L_z = \sum_{\alpha=1}^{N} l_{\alpha,z} = \sum_\alpha m_\alpha \omega (x_\alpha^2 + y_\alpha^2) = \omega \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = I_{zz}\omega_z = I\vec{\omega}$$

$(0\,0\,\omega)$

$$L_x = \sum_{\alpha=1}^{N} l_{\alpha,x} = \sum_\alpha m_\alpha (-\omega z_\alpha x_\alpha) = \omega \sum_\alpha m_\alpha (-x_\alpha z_\alpha) = I_{zx}\omega_z = I\vec{\omega}$$

$$\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

$$L_y = \sum_{\alpha=1}^{N} l_{\alpha,y} = \sum_\alpha m_\alpha (-\omega y_\alpha z_\alpha) = \omega \sum_\alpha m_\alpha (-y_\alpha z_\alpha) = I_{zy}\omega_z = I\vec{\omega}$$

~~moment of inertia~~
Product of inertia

→ in general $\vec{\omega}$ can be nonzero in all directions

⇒ then we also get $I_{xx}, I_{yy}, I_{xy}, I_{xz}, \ldots$

$$\Rightarrow \boxed{\vec{L} = I\,\vec{\omega}}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} \quad \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \quad \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

coordinate dependent — Inertia tensor ⇝ symmetric, positive definite

⇒ real eigenvalues

$$\begin{cases} I_{ii} = \sum_\alpha m_\alpha (j_\alpha^2 + k_\alpha^2) \\ I_{ij} = \sum_\alpha m_\alpha (-i_\alpha j_\alpha) = I_{ji} \end{cases}$$

KE:

if $\omega = (0,0,\omega)$

$$T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)\,\omega^2 = \frac{1}{2} I_{zz}\,\omega^2$$

$$\vec{v}_\alpha \cdot \vec{v}_\alpha = (-\omega y_\alpha)^2 + (\omega x_\alpha)^2 = \omega^2 (x_\alpha^2 + y_\alpha^2)$$

$\vec{v} = \omega \times \vec{r}$

Def.

Principal axes := axes of rotation s.t. if $\vec{\omega}$ is along the p.a. ⇒ $\vec{L} \parallel \vec{\omega}$ because $I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

p.a. always perpendicular

* If $\boxed{I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}}$ ⇒ $\hat{x}, \hat{y}, \hat{z}$ are principal axes.

Pf: $\vec{\omega} = \omega \hat{x}$: $\vec{L} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{xx}\omega \\ 0 \\ 0 \end{pmatrix} = \vec{\omega} I_{xx}$

⇒ $\vec{L} \parallel \vec{\omega}$

➤ $\vec{\omega} = \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$, $\vec{L} = I\vec{\omega} = I\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$ ⇒ $\vec{\omega} \parallel \vec{L}$ ⇒ $\boxed{I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}}$

Pf: $\vec{L} = I\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \omega I_{xx} \\ \omega I_{xy} \\ \omega I_{zx} \end{pmatrix} \equiv \begin{pmatrix} L \\ 0 \\ 0 \end{pmatrix}$

⇒ $I_{xy} = 0$ and $I_{zx} = 0$

⇒ repeat for $y, z$

- we can do coordinate transformation of $I$ to make it diagonal and then our basis vectors are the principal axes.

**Thm**

$\boxed{\forall \text{ rigid bodies and } \forall \text{ points in the rigid body, } \exists \text{ a set of 3 mutually orthogonal principal axes.}}$

Proof: $\vec{L} = I\vec{\omega} = \lambda\vec{\omega} = \lambda \mathbb{I}\vec{\omega}$ because we want $\vec{L} \parallel \vec{\omega}$

$\Rightarrow (I - \lambda \mathbb{1})\vec{\omega} = \vec{0}$ $\rightarrow$ $\vec{\omega}$ is eigenvector of $I$.

$\lambda = I'_{jj}$ - diagonal components of diagonal $I$.

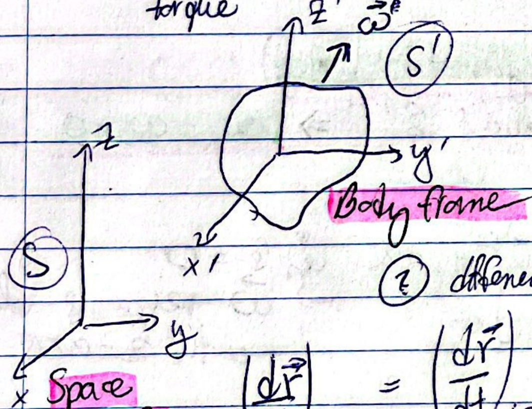
➤ If we have a symmetric body $\Rightarrow$ principal axes are usually axes of symmetry.

## Euler's equations for rigid body

- rotational version of Newton's 2nd law
- as seen by an observer rotating along with the rigid body

$$\boxed{\vec{\tau} = \dot{\vec{L}} + \vec{\omega} \times \vec{L}}$$

torque




$\vec{\omega} = (\omega'_1, \omega'_2, \omega'_3)$

$$I = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \rightsquigarrow x', y', z' \text{ are the principal axes}$$

(3) differential equation for $\omega'_1, \omega'_2, \omega'_3$

$$\underbrace{\left(\frac{d\vec{r}}{dt}\right)_{(S)}}_{\text{in } S} = \underbrace{\left(\frac{d\vec{r}}{dt}\right)_{(S')}}_{\text{derivative in } S'} + \underbrace{\vec{\omega} \times \vec{r}}_{\text{in } S'} \quad \text{holds for any vector}$$

$$\underbrace{\left(\frac{d\vec{L}}{dt}\right)_{(S)}}_{\tau} = \underbrace{\left(\frac{d\vec{L}}{dt}\right)_{(S')}}_{\dot{L}} + \vec{\omega} \times \vec{L}_{(\text{in } S')}$$

$\vec{r} = x'_i \hat{e}'_i$

$\dot{\vec{r}}_{(S')} = \dot{x}'_i \hat{e}'_i$

vs. $\dot{\vec{r}}_{(S)} = \dot{x}'_i \hat{e}'_i + x'_i \dot{\hat{e}}'_i$

$$\vec{L} = I\vec{\omega} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{pmatrix}$$

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega_1' & \omega_2' & \omega_3' \\ \lambda_1\omega_1' & \lambda_2\omega_2' & \lambda_3\omega_3' \end{vmatrix} = \hat{x}(\lambda_3\omega_2\omega_3 - \lambda_2\omega_2\omega_3) + \hat{y}(\lambda_1\omega_1\omega_3 - \lambda_3\omega_1\omega_3) + \hat{z}(\lambda_2\omega_1\omega_2 - \lambda_1\omega_1\omega_2)$$

$$= \hat{x}(\lambda_3 - \lambda_2)\omega_2\omega_3 + \hat{y}(\lambda_1 - \lambda_3)\omega_1\omega_3 + \hat{z}(\lambda_2 - \lambda_1)\omega_1\omega_2$$

$$\left(\frac{d\vec{L}}{dt}\right)_{(S)} = \frac{d}{dt}\begin{pmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{pmatrix} = \begin{pmatrix} \lambda_1\dot{\omega}_1 \\ \lambda_2\dot{\omega}_2 \\ \lambda_3\dot{\omega}_3 \end{pmatrix}$$

system of ODE's for components of $\omega$

$$\boxed{\begin{pmatrix} \tau_1' \\ \tau_2' \\ \tau_3' \end{pmatrix} = \tau' = \begin{pmatrix} \lambda_1\dot{\omega}_1' \\ \lambda_2\dot{\omega}_2' \\ \lambda_3\dot{\omega}_3' \end{pmatrix} + \begin{pmatrix} (\lambda_3-\lambda_2)\omega_2'\omega_3' \\ (\lambda_1-\lambda_3)\omega_1'\omega_3' \\ (\lambda_2-\lambda_1)\omega_2'\omega_1' \end{pmatrix}}$$

Euler equations for rigid body

$\hookrightarrow$ components in primed system ≙ body frame

real torque

(primed because written wrt primed coordinates

System of equations:

$$\boxed{\tau_i = \lambda_i\dot{\omega}_i + (\lambda_k - \lambda_j)\omega_j\omega_k}$$

☆ • $\vec{\tau} = 0$
• all $\lambda$'s different (asymmetric body)
• at $t=0$ rotating about $\hat{z}'$ $\Rightarrow$ $\omega_1 = \omega_2 = 0$

$$\Rightarrow \quad \left.\begin{aligned} 0 &= \lambda_1\dot{\omega}_1 + 0 \\ 0 &= \lambda_2\dot{\omega}_2 + 0 \\ 0 &= \lambda_3\dot{\omega}_3 + 0 \end{aligned}\right\} \Rightarrow \text{all } \dot{\omega}\text{'s} = 0$$

$\rightarrow \vec{\omega}$ remains aligned with $\hat{z}$-axis

☆ give the rigid body a small kick $\Rightarrow$ $\omega_1, \omega_2 \neq 0$
still $\vec{\tau} = 0$

$\omega_1, \omega_2$ small $\Rightarrow$ $\dot{\omega}_3$ small. $\Rightarrow$ treat $\omega_3$ as const.
$\Rightarrow$ we can decouple the system

$$-\lambda_1\dot{\omega}_1 = [(\lambda_3 - \lambda_2)\omega_3]\omega_2$$
$$-\lambda_2\dot{\omega}_2 = [(\lambda_1 - \lambda_3)\omega_3]\omega_1$$

treated as a constant

$$-\lambda_1 \ddot{\omega}_1 = [(\lambda_3-\lambda_2)\omega_3]\,\dot{\omega}_2 = \left[\frac{(\lambda_3-\lambda_2)\omega_3^2}{-\lambda_2}(\lambda_1-\lambda_3)\right]\omega_1$$

substitute from second eq.

$$\Rightarrow \ddot{\omega}_1 = \underbrace{\left[\frac{(\lambda_3-\lambda_2)\omega_3^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_2}\right]}_{=c}\omega_1 = c\,\omega_1$$

$$\boxed{\ddot{\omega}_1 - c\,\omega_1 = 0}$$

⇒ based on the sign of $c$, we can see whether the perturbation will be killed or exponentially increased.

$\lambda_i > 0$ always

$c<0$ ⇒ sin, cos (oscillations)

$\hookrightarrow (\lambda_3-\lambda_2)(\lambda_1-\lambda_3) < 0$

$\Rightarrow (\lambda_3-\lambda_2)(\lambda_3-\lambda_1) > 0$

$$\Rightarrow \begin{cases}\lambda_3 < \lambda_2\\ \lambda_3 < \lambda_1\end{cases} \quad \text{or} \quad \begin{cases}\lambda_3 > \lambda_2\\ \lambda_3 > \lambda_1\end{cases}$$

⇒ $\lambda_3$ is maximal or minimal eval ⇒ stable system

$c<0$ ⇒ $\exp(\alpha t)$, $\alpha > 0$ otherwise contradiction (if $\alpha<0$)

$\Rightarrow (\lambda_3-\lambda_2)(\lambda_3-\lambda_1) < 0$

$\lambda_2 < \lambda_3 < \lambda_1$ or $\lambda_1 < \lambda_3 < \lambda_2$

⇒ exponential increase in the perturbation

⇒ unstable system

★ axially symmetric body : $\lambda_1 = \lambda_2 \neq \lambda_3$

still $\vec{\tau} = 0$

$$\Rightarrow \quad 0 = \lambda_1\dot{\omega}_1 + (\lambda_3-\lambda_2)\,\omega_2\omega_3 \quad \Rightarrow \cancel{\lambda_1\dot{\omega}_1 = -(\lambda_3-\lambda_1)\omega_2\omega_3} = \Omega\omega_2$$

$$0 = \lambda_2\dot{\omega}_2 + (\lambda_1-\lambda_3)\,\omega_1\omega_3 \quad \Downarrow \quad \cancel{\lambda_2\dot{\omega}_2 = -(\lambda_1-\lambda_3}$$

$$0 = \lambda_3\dot{\omega}_3 + 0 \Rightarrow \boxed{\dot{\omega}_3 = 0}$$

$\Rightarrow$

$$\begin{cases}\lambda_1\dot{\omega}_1 = -(\lambda_3-\lambda_1)\,\omega_2\omega_3 \\ \lambda_1\dot{\omega}_2 = -(\lambda_1-\lambda_3)\,\omega_1\omega_3\end{cases}$$ using $\lambda_1=\lambda_2$

$$\Rightarrow \dot{\omega}_1 = \overbrace{-\frac{\lambda_3-\lambda_1}{\lambda_1}\,\omega_3}^{\Omega}\,\omega_2 = \Omega\omega_2$$

$$\dot{\omega}_2 = -\omega_1\omega_3\frac{(\lambda_1-\lambda_3)}{\lambda_1} = -\Omega\omega_1$$

Let $\eta = \omega_1 + i\omega_2 \Rightarrow \boxed{\dot{\eta} = -i\Omega\eta}$

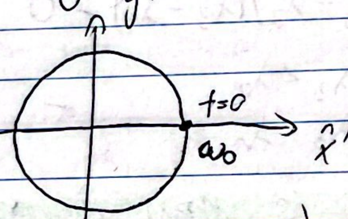
$$\Rightarrow \eta = \eta_0 e^{-i\Omega t}$$

$$\eta(t=0) = \{\underbrace{\overset{\in\mathbb{R}}{\omega_0}}_{\omega_1(0)}\} \rightarrow \eta = \omega_0\cos(\Omega t) - i\omega_0\sin(\Omega t)$$

$$\Rightarrow \vec{\omega} = \begin{pmatrix}\omega_0\cos(\Omega t) \\ -\omega_0\sin(\Omega t) \\ \omega_3\end{pmatrix} = \begin{pmatrix}\omega_1(t) \\ \omega_2(t) \\ \omega_3\end{pmatrix}$$

precession $\Rightarrow$ oscillatory behaviour, $\omega_3$ remains the same

$\Rightarrow$

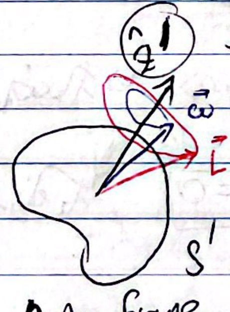



$$\vec{L} = \begin{pmatrix}\lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3\end{pmatrix} = \begin{pmatrix}\lambda_1\omega_0\cos\Omega t \\ -\lambda_2\omega_0\sin\Omega t \\ \lambda_3\omega_3\end{pmatrix}$$

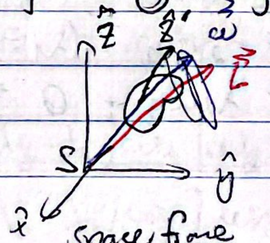
$\Rightarrow \vec{L}$ will also draw a circle in $x'y'$ plane with same angular freq. $\Omega$ as $\omega$

$\Rightarrow \vec{L}$ and $\vec{\omega}$ precess about $\hat{z}'$-axis (in the body frame)

but in inertial frame = space frame, I cannot expect ~~...~~ $\vec{L}$ to change because $0 = \vec{\tau} = \left(\frac{d\vec{L}}{dt}\right)_{|S}$




Body frame




Space frame

body is rotating

$\Downarrow$

$\vec{L}$ fixed and $\hat{z}'$ and $\omega$ rotate (precess about $\vec{L}$)

2e 2e

$t \to u : 100\,000 = 10^5$

$b \to d : 1000 = 10^3$

$p'tp: 10^{10} \cdot 10^3 = 10^{13}$

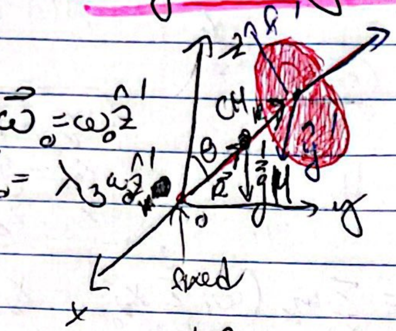
$n'/n: 10^5 \cdot 10^6 = 10^{11}$

$p^{\text{hi}}$ 440 GeV

$h' \sim 208$

1280 → 1.3 TeV

# ☆ Rigid body : disk on a rod




$\vec{\omega}_0 = \omega_0 \hat{z}'$

$\Rightarrow \vec{L}_0 = \lambda_3 \omega_0 \hat{z}'$

$\vec{g} \neq 0 \Rightarrow \vec{L} \neq$ const.

$\vec{\tau}$ = small

$\vec{\tau} = \vec{R} \times \vec{F}_G = \vec{R} \times M\vec{g}$

$\dot{\vec{L}}\big|_{(S)} = \vec{\tau}$

$\vec{R} \parallel \hat{z}'$

$\Rightarrow \vec{\tau} \perp \hat{z}'$

because $\vec{A} \times \vec{B} \perp \vec{A}$, $\perp \vec{B}$

$\Rightarrow \dot{\vec{L}}\big|_{(S)} = \vec{R} \times M\vec{g}$

$|\vec{\tau}| = RMg\sin(\pi - \theta) = RMg\sin\theta$

$$\frac{d\vec{L}}{dt}\bigg|_{(S)} = \frac{d}{dt}(\lambda_3 \omega \hat{z}') = \vec{\tau} = \vec{R} \times M\vec{g} = RMg(-\hat{z}' \times \hat{z}) = RMg(\hat{z} \times \hat{z}')$$

$\underline{\tau \text{ is weak}} \Rightarrow \omega_1, \omega_2$ small ($\vec{L}$ remains roughly along $\hat{z}'$)

$\Rightarrow$ value of $\vec{\omega}$ won't change a lot, only the direction

$\Rightarrow \frac{d}{dt}(\lambda_3 \omega \hat{z}') = \lambda_3 \omega \dot{\hat{z}}' + \lambda_3 \dot{\omega} \hat{z}'$, $\lambda_3 \dot{\omega} \hat{z}' \to 0$ ~ proof in the notes and below *

$$\boxed{\Rightarrow \dot{\hat{z}}' = \underbrace{+\frac{Rmg}{\lambda_3 \omega}(\hat{z} \times \hat{z}')}_{\equiv \vec{\Omega}} = \vec{\Omega} \times \hat{z}'}$$

$\Rightarrow \hat{z}'$ rotates with angular velocity $\Omega$

because $\dot{\vec{r}} = \vec{\omega} \times \vec{r}$

> $\Rightarrow$ When a rigid body like this spins about $\hat{z}'$ axis, the $\hat{z}'$-axis precesses about the $\hat{z}$-axis

* proof of $\frac{d\vec{L}}{dt} = \lambda_3 \omega \dot{\hat{z}}' + \lambda_3 \dot{\omega} \hat{z}'$

$$\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 \dot{\omega}_1 \\ \lambda_2 \dot{\omega}_2 \\ \lambda_3 \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} (\lambda_3 - \lambda_2)\,\omega_1 \\ (\lambda_1 - \lambda_3)\,\omega_2 \\ (\lambda_2 - \lambda_1)\,\omega_3 \end{pmatrix}$$

$\hookrightarrow \tau_3 = 0$ because $\tau \parallel \hat{z}$ and $\lambda_2 = \lambda_1 \Rightarrow \lambda_2 - \lambda_1 = 0$

$\Rightarrow \lambda_3 \dot{\omega}_3 = 0 \Rightarrow \dot{\omega}_3 = 0$

$$\Rightarrow \frac{d\vec{L}}{dt} = \lambda_3 \omega \dot{\hat{z}}'$$

derivation

$\vec{L} = I\vec{\omega}$

$A\times(B\times A) = A^2\vec{B} - \vec{A}(\vec{A}\cdot\vec{B})$

$$\vec{L} = \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \times (\vec{\omega}\times\vec{r}_\alpha) =$$

$x_{\alpha_1} = x_\alpha$, $x_{\alpha_2} = y_\alpha$, $x_{\alpha_3} = z_\alpha$

$$= \sum_{\alpha=1}^{N} m_\alpha \left(r_\alpha^2\vec{\omega} - \vec{r}_\alpha(\vec{r}_\alpha\cdot\vec{\omega})\right)$$

$$L_i = \sum_\alpha m_\alpha \left(r_\alpha^2\omega_i - (\vec{r}_\alpha\cdot\vec{\omega})r_i\right) = \sum_\alpha m_\alpha \left(\omega_i \sum_{j=1}^{3} x_{\alpha_j}^2 - x_{\alpha_i}\sum_{k=1}^{3} x_{\alpha_k}\omega_k\right)$$

$$= \sum_{k=1}^{3} \underbrace{\boxed{\sum_\alpha m_\alpha \left[\delta_{ij}\sum_{j=1}^{3} x_{\alpha_j}^2 - x_{\alpha_i}\cdot x_{\alpha_k}\right]}}_{I_{ik}} \omega_k$$

$$\Rightarrow L_i = \sum_{k=1}^{3} I_{ik}\,\omega_k \quad \text{with} \quad \boxed{I_{ik} = \sum_\alpha m_\alpha\left[\delta_{ik}\sum_{j=1}^{3} x_{\alpha_j}^2 - x_{\alpha_i}x_{\alpha_k}\right]}$$

$$\Rightarrow \boxed{\vec{L} = I\vec{\omega}}$$

e.g. $I_{11} = \sum_\alpha m_\alpha\left[x_1^2+x_2^2+x_3^2 - x_1^2\right] = \sum_\alpha m_\alpha (x_2^2+x_3^2)$ ✓

$I_{12} = \sum_\alpha m_\alpha (0 - x_{\alpha_1} x_{\alpha_2}) = -\sum_\alpha m_\alpha x_{\alpha_1} x_{\alpha_2}$ ✓

as found before

rotational energy

$\xrightarrow{\text{rotation}}$, rigid body $\Rightarrow \vec{\omega}$ for all $\alpha$

$$T_{rot} = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha |\vec{v}_\alpha|^2 = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha (\vec{\omega}\times\vec{r}_\alpha)^2 \overset{*}{=}$$

$$= \frac{1}{2}\sum_{ij} \omega_i I_{ij}\omega_j = \frac{1}{2}\underset{\omega^T}{\vec{\omega}}\cdot I\cdot\omega =$$

$$\boxed{T_{rot} = \frac{1}{2}\omega^T I\omega}$$

$$= (\omega_1\ \omega_2\ \omega_3)(3\times 3 \text{ matrix})\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \text{scalar}$$

(for diagonal $I \Rightarrow \omega^T I\omega = I\omega^2$

$(\vec{A}\times\vec{B})^2 = (\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B}) - (\vec{A}\cdot\vec{B})^2$

$$\overset{*}{=} \frac{1}{2}\sum_\alpha m_\alpha\left[(\vec{\omega}\cdot\vec{\omega})(\vec{r}_\alpha\cdot\vec{r}_\alpha) - (\vec{\omega}\cdot\vec{r}_\alpha)^2\right] =$$

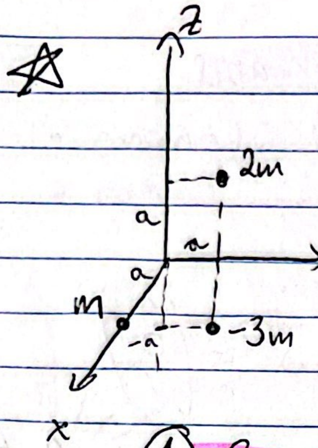
$$= \frac{1}{2}\sum_\alpha m_\alpha\left[\left(\sum_{j=1}^{3}\omega_j^2\right)\left(\sum_{k=1}^{3}x_{\alpha_k}^2\right) - \left(\sum_{i=1}^{3}\omega_i x_{\alpha_i}\right)\left(\sum_{j=1}^{3}\omega_j x_{\alpha_j}\right)\right] =$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \sum_{ij}\omega_i\omega_j\left[\delta_{ij}\sum_{k=1}^{3}x_{\alpha_k}^2 - x_{\alpha_i}x_{\alpha_j}\right] =$$

$$= \frac{1}{2}\sum_{ij} \underbrace{\boxed{\sum_\alpha m_\alpha \left[\delta_{ij}\sum_{k=1}^{3}x_{\alpha_k}^2 - x_{\alpha_i}x_{\alpha_j}\right]}}_{I_{ij}} \omega_i\omega_j$$

$$\Rightarrow \boxed{T_{rot} = \frac{1}{2}\sum_{ij} I_{ij}\,\omega_i\omega_j}$$

★




$m : (a,0,0)$
$2m : (0,a,a)$
$3m : (0,a,-a)$
rigid body ⇒ fixed distances

? principal moments ? principal axes

① Compute $I$ wrt given reference frame

$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2) = m(0+0) + 2m(a^2+a^2) + 3m(a^2+a^2) =$$
$$= 4ma^2 + 6ma^2 = \underline{10ma^2}$$
$$I_{yy} = \sum_\alpha m_\alpha (x_\alpha^2 + z_\alpha^2) = m(a^2) + 2m(0+a^2) + 3ma^2 = \underline{6ma^2}$$
$$I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = ma^2 + 2ma^2 + 3ma^2 = \underline{6ma^2}$$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha = -m\cdot 0 - 2m\cdot 0 - 3m\cdot 0 = \underline{0}$$

$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha = \underline{0}$$
$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha = -(0 + 2ma^2 - 3ma^2) = \underline{ma^2}$$

$$\Rightarrow I = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} ma^2 \quad c$$

② Diagonalise $I$: Find eigenvalues = principal moments $I_{11}, I_{22}, I_{33}$

$$0 = \det(I - \lambda \mathbb{1}) = \begin{vmatrix} 10c-\lambda & 0 & 0 \\ 0 & 6c-\lambda & 1c \\ 0 & 1c & 6c-\lambda \end{vmatrix} = (10c-\lambda)\left[(6c-\lambda)^2 - 1c^2\right]$$
$$\Rightarrow 0 = (10c-\lambda)(\lambda^2 - 12c\lambda + (36-1)c^2)$$

$$\Rightarrow 10c - \lambda = 0 \Rightarrow \lambda_1 = 10ma^2$$

or

$$\lambda^2 - 12\lambda ma^2 + 35m^2a^4 = 0 \Rightarrow \lambda_{1,2} = \frac{12ma^2 \mp \sqrt{144c^2 - 4\cdot 35c^2}}{2}$$
$$= 6ma^2 \mp ma^2\sqrt{36-35} = 6ma^2 \mp ma^2$$

$$\lambda_2 = 5ma^2$$
$$\lambda_3 = 7ma^2$$
principal moments

③ Find eigenvectors = principal axes

$\underline{\lambda_1 = 10ma^2}$

$$(I - \lambda_1 \mathbb{1})\vec{\omega}^{(1)} = 0$$

$$\left[\begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix} - \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}\right] ma^2 \, \omega^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -15 & 0 \end{array}\right)$$

$$\Rightarrow \begin{rcases} \omega_3 = 4\omega_2 \\ \omega_3 = 0 \end{rcases} \Rightarrow \omega_2 = 0, \; \omega_3 = 0$$

$$\omega_1 = 1 \Rightarrow \omega^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

principal axis 1 = $\underline{\underline{x\text{-axis}}}$

$\underline{\lambda_2 = 5ma^2}$

$$\left(\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \Rightarrow \begin{aligned} \omega_1 &= 0 \\ \omega_2 &= -\omega_3 \end{aligned} \Rightarrow \omega^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$\underline{\lambda_3 = 7ma^2}$

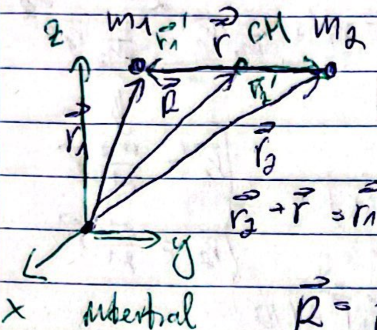
$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \Rightarrow \begin{aligned} \omega_1 &= 0 \\ \omega_2 &= \omega_3 \end{aligned} \Rightarrow \omega^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$\Rightarrow$ the principal axes are along $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

along the 3 masses

# Central conservative forces
## (Kepler's laws)

- conservative forces
- central forces - only depend on relative positive of the masses




$$\vec{F}_{12} = f(|\vec{r}_1 - \vec{r}_2|)\,\hat{r}_{12} = f(\vec{r})\,\hat{r}$$

$$\vec{r}_2 + \vec{r} = \vec{r}_1 \;=\; \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\vec{R} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2) \quad , M = m_1 + m_2$$

$\boxed{\text{use } (\vec{R}, \vec{r}) \text{ instead of } (\vec{r}_1, \vec{r}_2)}$ $\Rightarrow$ simplifies the problem

$$L = T - U = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - U(r) = L(\vec{r}, \vec{R}, \dot{\vec{r}}, \dot{\vec{R}})$$

$\vec{r}_1 = f_1(\vec{R}, \vec{r})$

$\vec{r}_2 = f_2(\vec{R}, \vec{r})$

$$\begin{cases} M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 \\ \vec{r} = \vec{r}_1 - \vec{r}_2 \end{cases} \Rightarrow \begin{cases} \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r} \\ \vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}} \\ \dot{\vec{r}}_1 = \dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}} \end{cases}$$

$$\Rightarrow \boxed{L = \underbrace{\frac{1}{2}M\dot{\vec{R}}^2}_{\text{kinetic energy of body of mass } M \text{ at CM}} + \underbrace{\frac{1}{2}\mu\dot{\vec{r}}^{\,2}}_{\text{reduced mass KE}} + U(r) \quad , \underbrace{\mu}_{\text{reduced mass}} = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}}$$

if $\underline{m_1 >> m_2}$: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \approx \frac{m_1 m_2}{m_1} = m_2$

$M = m_1 + m_2 \approx \underline{m_1}$

$\dot{\vec{R}} = \text{const.}$ - moves with const. speed
because no "external forces" = no forces on CM

easy to deal with $\Rightarrow$ reduced number of "difficult" coordinates

coordinates

$$\left.\begin{matrix} \vec{r}_1 - 3 \\ \vec{r}_2 - 3 \end{matrix}\right\} 6 \begin{cases} 3 - \vec{R} \\ 3 - \vec{r} \end{cases}$$

simple motion at const. velocity

coordinate system attached to $\vec{R}$ is inertial

we can move to a coordinate system centered at CM

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1'^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2'^{\,2} - U(r)$$

$\rightarrow$ use it.

again rewrite to $\vec{r}$ and $\vec{R}$. But now in coordinate system at CM $\Rightarrow \vec{R} = 0 = m_1\vec{r}_1' + m_2\vec{r}_2'$

$\hookrightarrow \vec{r}_2' = -\frac{m_1}{m_2}\vec{r}_1'$

and I have $\vec{r} = \vec{r}_1' - \vec{r}_2' = \vec{r}_1' + \frac{m_1}{m_2}\vec{r}_1' = \frac{M}{m_2}\vec{r}_1'$

$$\Rightarrow L = \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2 - U(r)$$

$$\left\{\begin{array}{l} \vec{r}_1' = \frac{m_2}{M}\vec{r} \\ \vec{r}_2' = \frac{m_1}{M}\vec{r} \end{array}\right.$$

$$L = \frac{1}{2}m_1\frac{m_2^2}{M^2}\dot{\vec{r}}^2 + \frac{1}{2}m_2\frac{m_1^2}{M^2}\dot{\vec{r}}^2 - U(r)$$

$$= \frac{1}{2}\frac{m_1 m_2}{M}\frac{m_2}{M}\dot{\vec{r}}^2 + \frac{1}{2}\frac{m_2 m_1}{M}\frac{m_1}{M}\dot{\vec{r}}^2 - U(r)$$

$$= \frac{1}{2}\frac{m_2}{M}\mu\dot{\vec{r}}^2 + \frac{1}{2}\mu\frac{m_1}{M}\dot{\vec{r}}^2 - U(r)$$

$$= \frac{1}{2}\mu\dot{\vec{r}}^2\frac{m_2+m_1}{M} - U(r)$$

$$\Rightarrow \boxed{L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)}$$

☆ If $m_1 \gg m_2$, then $\left.\begin{array}{l}\mu \approx m_2 \\ M \approx m_1 \\ \vec{R} \approx \vec{r}_1\end{array}\right\} \Rightarrow$ 1 body problem

## Conserved quantities

→ **momentum**: $\vec{P} = 0$ because $F_{ext} = 0$

total momentum: $\vec{P} = M\dot{\vec{R}}$ — $U$ indep. of $\vec{R}$

→ **angular momentum**: $\dot{\vec{L}} = \vec{\tau} = 0$ because the system is isolated (internal forces cancel out by Newton's 3rd law)

"total angular momentum"

→ **energy**: $E = H$

$\hookrightarrow \frac{\partial L}{\partial t} = 0$

$\hookrightarrow$ time-indep. $L$ and $q$

$\hookrightarrow \vec{L}\cdot\vec{r} = (\vec{r}\times\vec{p})\cdot\vec{r} = 0$

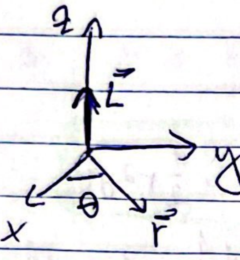
because $\vec{r}\times\vec{p} \perp \vec{r}$

$\hookrightarrow \vec{L} = \text{const.}$ and $\vec{r}\perp\vec{L} \Rightarrow \vec{r} \perp$ constant vector

$\Rightarrow \vec{r}$ is in a const. plane $\perp \vec{L}$

3D → 2D $\Rightarrow$ motion in plane

$\Rightarrow$ use polar / cylindrical coordinates




$\vec r = (r\cos\theta, r\sin\theta, 0)$

$\vec L \parallel \hat z$

$\dot{\vec r}^2 = \dot x^2 + \dot y^2 = (\dot r\cos\theta - r\sin\theta\,\dot\theta)^2 + (\dot r\sin\theta + r\cos\theta\,\dot\theta)^2 =$

$= \dot r^2 + r^2\dot\theta^2 - 2r\dot r\dot\theta\cos\sin + 2r\dot r\dot\theta\cos\sin$

$= \dot r^2 + r^2\dot\theta^2$

$\Rightarrow \boxed{\mathcal{L}_{CM} = \frac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) - U(r)}$

Lagrangian in CM frame — only 2 degrees of freedom!

$\underbrace{\frac{\partial L}{\partial \theta}}_{=0} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot\theta} = \text{const.} = p_\theta$

$\Rightarrow \boxed{p_\theta = \mu r^2\dot\theta = \text{constant}} \equiv l$

$\frac{d}{dt}(\mu r^2\dot\theta) = \mu\underbrace{(2\dot r\dot\theta + r\ddot\theta)}_{a_\theta} r = 0$ angular momentum about CM $\Rightarrow$ also conserved

$\Rightarrow \boxed{a_\theta = 0} \Rightarrow F_\theta = 0$

$l = \vec L = \mu\vec r \times \dot{\vec r} = \mu\begin{pmatrix}\hat x & \hat y & \hat z \\ r\cos\theta & r\sin\theta & 0 \\ \dot x & \dot y & 0\end{pmatrix} = 0\hat x + 0\hat y + \mu(r\dot y\cos\theta - \dot x r\sin\theta)\hat z$

$= \mu[r\overset{\sin\theta}{\cancel{\cos\theta}}(\dot r\cos\theta - r\dot\theta\sin\theta) - r\overset{\cos\theta}{\cancel{\sin\theta}}(\dot r\sin\theta + r\dot\theta\cos\theta)]\hat z$

$\cancel{= (r\dot r\cos^2\theta\sin\theta - r^2\dot\theta\cos\theta\sin^2\theta - r\dot r\sin^2\theta - r^2\dot\theta\sin\theta\cos\theta)\hat z}$

$= \mu(-r\dot r\sin\theta\cos\theta + r^2\dot\theta\sin^2\theta + r\dot r\cos\theta\sin\theta + r^2\dot\theta\cos^2\theta)\hat z$

$= \mu r^2\dot\theta\,\hat z$

in general

$\boxed{\vec r = r\hat r \rightsquigarrow \vec a \equiv \ddot{\vec r} = \overbrace{(\ddot r - r\dot\theta^2)}^{a_r}\hat r + \underbrace{(r\ddot\theta + 2\dot r\dot\theta)}_{a_\theta}\hat\theta}$

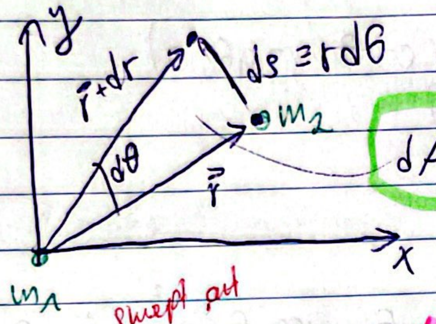
$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} = 0$

$\mu r\dot\theta^2 - U'(r) - \mu\ddot r = 0$

$\Rightarrow \mu\ddot r - \mu r\dot\theta^2 + U'(r) = 0$ $\quad a_\theta = 0 \Rightarrow a = a_r$

$\Rightarrow \boxed{-\frac{dU}{dr} = \mu\underbrace{(\ddot r - r\dot\theta^2)}_{a_r} = \mu a}$ Newton's law ($2^{nd}$)

# Kepler's 2nd law




$$\boxed{dA = \frac{1}{2} r (r\,d\theta) = \frac{1}{2} r^2 d\theta}$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$\boxed{\frac{dA}{dt} = \frac{l}{2\mu} = \text{const.}}$$

$\Leftarrow$ area covered (swept out) by $\vec{r}$ vector in a given time is const. ~~always~~

## Equations for orbit $r = r(\theta)$

$$\mu \ddot{r} = -\frac{dU}{dr} + \underbrace{\mu r \dot{\theta}^2}_{= \mu r \left(\frac{l}{\mu r^2}\right)^2 = \frac{l^2}{\mu r^3}}$$

angular momentum

$$\Rightarrow \mu \ddot{r} = -\frac{dU}{dr} + \frac{l^2}{\mu r^3} = -\frac{dU}{dr} - \frac{d}{dr}\left(\frac{l^2}{2\mu r^2}\right)$$

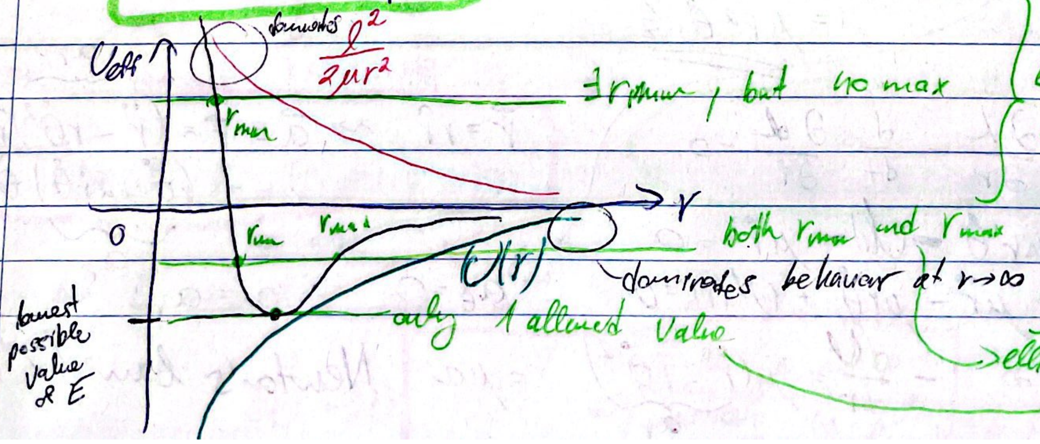
$$\Rightarrow \mu \ddot{r} = -\frac{d}{dr}\underbrace{\left(U + \frac{l^2}{2\mu r^2}\right)}_{U_{eff}} = -\frac{d}{dr} U_{eff}(r)$$

$U_{eff}$ = effective potential

$$\boxed{\mu \ddot{r} = -\frac{d}{dr} U_{eff}(r)}$$

Consider gravitational potential (force) $\propto \frac{1}{r}$

$$\boxed{U(r) = -\frac{G m_1 m_2}{r}} = -\frac{\gamma}{r}$$




unbounded
$E > 0$
$\Rightarrow$ open orbit
$\Rightarrow$ parabola, hyperbola

closed orbit
$\Downarrow$
ellipse / circle

## ⑦ Types of orbits based on energy

$$E = \underbrace{\frac{\mu}{2}(\dot r^2 + r^2\dot\theta^2)}_{KE} + U(r) = \frac{\mu}{2}\dot r^2 + \frac{\mu}{2} r^2\left(\frac{l}{\mu r^2}\right)^2 + U(r) =$$

$$= \frac{\mu}{2}\dot r^2 + \frac{l^2}{2\mu r^2} + U(r) = \boxed{\underbrace{\frac{\mu}{2}\dot r^2}_{\geq 0} + V_{eff}} \geq V_{eff}$$

$$\frac{df}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt} = \dot\theta \frac{df}{d\theta} = \frac{l}{\mu r^2}\frac{d}{d\theta}$$

$$\Rightarrow \dot r = \frac{l}{\mu r^2}\frac{dr}{d\theta}$$ $\boxed{\text{Let } u := \frac{1}{r}}$

$$\Rightarrow \ddot r = \frac{d}{dt}\frac{l}{\mu r^2}\frac{dr}{d\theta} = \left(\frac{d}{d\theta}\frac{dr}{dt}\right)\frac{l}{\mu r^2} - \frac{dr}{d\theta}\frac{d}{dt}\frac{l}{\mu}u^2 =$$

$$= \frac{d}{d\theta}\left(\frac{l}{\mu r^2}\frac{dr}{d\theta}\right)\frac{l}{\mu r^2} - \frac{dr}{d\theta}\frac{l}{\mu}2u\dot u$$

$$\Rightarrow \boxed{r(\theta) = \frac{c}{1+\varepsilon\cos\theta}} \quad , \quad \boxed{\begin{matrix} c = \dfrac{l^2}{\mu (G m_1 m_2)} \\ \varepsilon = \text{const.} \cdot c \end{matrix}}$$

eccentricity

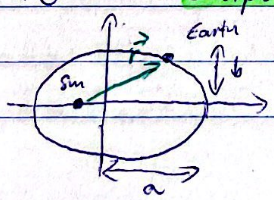
**$\varepsilon \geq 1$, unbounded orbits**

- $\varepsilon = 1$, $E = 0$ $\Rightarrow$ parabola
- $\varepsilon > 1$, $E > 0$ $\Rightarrow$ hyperbola

**$\varepsilon < 1$, bounded orbits**

$1 + (\cos\theta)\cdot\varepsilon$:
- max: $1+\varepsilon \Rightarrow r_{min} = \frac{c}{1+\varepsilon}$
- min: $1-\varepsilon \Rightarrow r_{max} = \frac{c}{1-\varepsilon}$

- (if $\varepsilon = 0 \Rightarrow r_{max} = r_{min} = r$) $\Rightarrow$ circle, $E = E_{min}$
- $0 < \varepsilon < 1$, $E_{min} < E < 0$ $\Rightarrow$ ellipse




$$a = \frac{c}{1-\varepsilon^2}$$
$$b = \frac{c}{\sqrt{1-\varepsilon^2}}$$
$$\varepsilon = \sqrt{1-\left(\frac{b}{a}\right)^2}$$

# Kepler's 3rd law

$\gamma = G m_1 m_2$

$c = \frac{\ell^2}{\mu \gamma}$

$a = \frac{c}{1-\varepsilon^2}$

$b = a\sqrt{1-\varepsilon^2}$

$$\frac{dA}{dt} = \frac{\ell}{2\mu} \Rightarrow dt = \frac{2\mu}{\ell}\, dA$$

For ellipse $A = \pi a b$

$$\tau = \int_0^\tau dt = \int \frac{2\mu}{\ell}\, dA = \frac{2\mu}{\ell}\int dA = \frac{2\mu}{\ell}\pi a b$$

$$\Rightarrow \boxed{\tau^2 = \frac{4\pi^2}{G}\,\frac{a^3}{M}}$$ Kepler's 3rd law

$\hookrightarrow M = m_1 + m_2$

② $E(\varepsilon) = ?$

$E = \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r)$

const $\rightsquigarrow$ $\dot{r} = 0 \Leftrightarrow E = V_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{\mu r_{min}^2}$

$$\begin{cases} r_{min} = \frac{c}{1+\varepsilon} \\ c = \frac{\ell^2}{\mu\gamma} \end{cases} \Rightarrow E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)$$

$$\rightsquigarrow \boxed{E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)}$$

$$\begin{cases} \varepsilon = 1 \rightarrow E = 0 \\ \varepsilon > 1 \rightarrow E > 0 \\ \varepsilon < 1 \rightarrow E < 0 \end{cases}$$

| eccentricity | energy | orbit |
|---|---|---|
| $\varepsilon > 1$ | $E > 0$ | hyperbola |
| $\varepsilon = 1$ | $E = 0$ | parabola |
| $0 < \varepsilon < 1$ | $E < 0$ | ellipse |
| $\varepsilon = 0$ | $E < 0$ | circle |

# Special Relativity

**Axioms:**

1. The laws of physics are the same in all inertial reference frames } Galilean principle of relativity
2. velocity of light in vacuum is the same in all inertial reference frames

To ~~do~~

## Galilean → Lorentz transformations

### Failure of Galilean transformation in electromagnetism

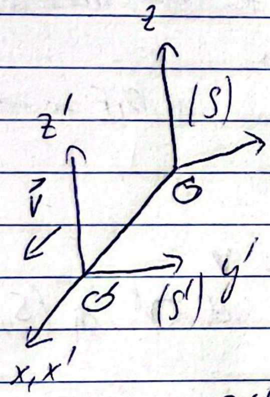
Maxwell's equations : $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t}$

$\nabla^2 \vec{B} = \underbrace{\mu_0 \varepsilon_0}_{1/c^2} \frac{\partial \vec{B}}{\partial t}$

↳ for each component - wave eq.

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}} \quad \text{in } (S)$$

⇝ solution: $\psi = \psi_0 e^{ik(x-ct)}$




$$\begin{cases} x' = x - vt \\ t' = t \\ y' = y \\ z' = z \end{cases}$$

move this to primed frame

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial \psi}{\partial t'} \underbrace{\frac{\partial t'}{\partial x}}_{0} = \frac{\partial \psi}{\partial x'} \Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x'^2}$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial \psi}{\partial t'} \frac{\partial t'}{\partial t} = -v \frac{\partial \psi}{\partial x'} + \frac{\partial \psi}{\partial t'}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t'} - v \frac{\partial \psi}{\partial x'}\right) = \frac{\partial}{\partial t'}\left(\frac{\partial \psi}{\partial t'} - v \frac{\partial \psi}{\partial x'}\right)$$

$$- v \frac{\partial}{\partial x'}\left(\frac{\partial \psi}{\partial t'} - v \frac{\partial \psi}{\partial x'}\right) =$$

$$= \frac{\partial^2 \psi}{\partial t'^2} - 2v \frac{\partial^2 \psi}{\partial t' \partial x'} + v^2 \frac{\partial^2 \psi}{\partial x'^2}$$

⇒ Wave eq.: $$\boxed{\frac{\partial^2 \psi}{\partial x'^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t'^2} - \frac{2v}{c^2} \frac{\partial^2 \psi}{\partial t' \partial x'} + \frac{v^2}{c^2} \frac{\partial^2 \psi}{\partial x'^2}} \quad \text{in } (S')$$

What if solved by wave eq. sol? with different speed

$$\psi = \psi_0 e^{ik(x'-v't')}, \quad v' = c - V$$

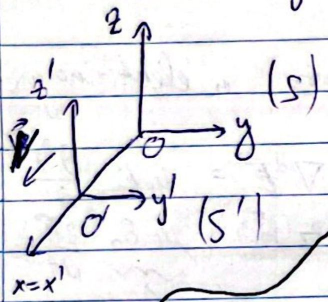
⇒ but then speed of light isn't the same in all frames !!

## Lorentz transformations for contravariant vector $x^\mu$

• $S'$ moving in $x=x'$ axis

at $t=0$: • $O=O'$ (origins same)

• light beam emitted



• using 2nd axiom:

(S) $x^2+y^2+z^2=c^2t^2$

(S') $x'^2+y'^2+z'^2=c^2t'^2$

• transformations must be linear because otherwise uniform motion in one frame wouldn't be uniform in the other (contradiction with 1st axiom)

$y^2+z^2=y'^2+z'^2$

$\Rightarrow \boxed{x^2-c^2t^2=x'^2-c^2t'^2}$

$$\begin{cases} x'=Ax+Bt \\ t'=Dx+Ft \\ y'=y \\ z'=z \end{cases}, \quad A,B,D,F = \text{const.}$$

$x^2-c^2t^2=(Ax+Bt)^2-c^2(Dx+Ft)^2$

$x^2-c^2t^2=A^2x^2+2ABxt+B^2t^2$
$\quad -c^2D^2x^2-2c^2DAxt-c^2A^2t^2$

• velocity of $O'$ as seen by $O$ is $V$.

$0=Ax+Bt$ — coordinates of $O'$ in (S)

$Ax=-Bt$

$\frac{\Delta x}{\Delta t}=\frac{x}{t}=\boxed{-\frac{B}{A}=V}$ — velocity of $O'$ in (S)

• velocity of $O$ as seen by $O'$ is $-V$

$\begin{cases} x'=0+Bt \\ t'=0+Ft \end{cases}$ — coordinates of $O$ as viewed from (S')

$\Rightarrow \frac{x'}{t'}=\boxed{\frac{B}{F}=-V}$ — velocity of in (S')

$\Rightarrow \frac{B}{A}=\frac{B}{F} \Rightarrow \boxed{A=F}$

↳ polynomials in $x,t$ need to be equal at all $x,t$

$\Rightarrow x^2=A^2x^2-c^2D^2x^2 \Rightarrow A^2-1=c^2D^2$

$0=2ABxt-2c^2DAxt \Rightarrow AB=c^2DA \Rightarrow B=c^2D$

$-c^2t^2=B^2t^2-c^2A^2t^2 \Rightarrow B^2=c^2(A^2-1)$

} solve

$$\Rightarrow \begin{cases} A = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \\ B = \dfrac{-v}{\sqrt{1-\frac{v^2}{c^2}}} = -v\gamma \\ D = \dfrac{-v/c^2}{\sqrt{1-\frac{v^2}{c^2}}} = -\frac{v}{c^2}\gamma \end{cases}$$

$\boxed{\text{Non-relativistic}: \gamma \approx 1}$

$$\boxed{\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$

$\beta = \frac{v}{c}$

$$\Rightarrow \quad x' = \gamma x - v\gamma t = \gamma(x - vt) = \gamma(x - \beta(ct))$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma t - \frac{v}{c^2}\gamma x \quad \rightsquigarrow \quad ct' = \gamma\left(ct - \frac{v}{c}x\right) = \gamma(ct - \beta x)$$

coordinates: $(x_0, x_1, x_2, x_3) = (ct, x, y, z)$

$$\Rightarrow \boxed{\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}}$$

$$\boxed{\begin{aligned} x_0' &= ct' = \gamma((ct) - \beta x) \\ x_1' &= x' = \gamma(x - \beta(ct)) \end{aligned}}$$

can be generalised for $\beta$ along any direction

## Four vectors

$x^\mu = (ct, \vec{r})$

contravariant $x^\mu = (x^0, x^1, x^2, x^3)$

covariant $x_\mu = (x^0, -x^1, -x^2, -x^3)$

Euclidean space vs. Minkowski spacetime

$\vec{x} \cdot \vec{x} = x_1^2 + x_2^2 + x_3^2$

$$x^\mu x_\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$
$$= x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3$$
$$= x^0 x^0 - x^1 x^1 - x^2 x^2 - x^3 x^3$$

distance: $\Delta\vec{x} \cdot \Delta\vec{x} > 0$

between $(ct, \vec{x})$ and $(ct', \vec{x}')$

$$\Rightarrow \Delta t = t - t', \quad \Delta\vec{x} = \vec{x} - \vec{x}'$$

$$\rightarrow \boxed{(\Delta s)^2 = (c\,\Delta t)^2 - \Delta\vec{x} \cdot \Delta\vec{x}}$$

- $> 0$ communication - via ... (massive)
- $= 0$ - via light
- $< 0$ - not possible → causally disconnected events

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow \eta^{\mu\nu} = \eta_{\mu\nu}^{-1}$$

$\boxed{\text{"metric" tensor to go between covariant and contravariant}}$

$$\begin{cases} x^\mu = (ct, \vec{x}) & \text{contravariant vector} \\ x_\mu = (ct, -\vec{x}) & \text{covariant vector} \end{cases}$$

(meaning holds for all vectors)

$$\boxed{x_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\nu} \rightarrow \begin{cases} \mu=0: & \sum_\nu \eta_{0\nu} x^\nu = \eta_{00} x^0 = ct = x_0 \\ \mu=1,2,3: & \sum_\mu \eta_{1\nu} x^\nu = -x^1 = x_1 \end{cases}$$

scalar product $\vec{x} \cdot \vec{x}$

$$\Rightarrow \boxed{x^\mu x_\mu = (\eta_{\mu\nu} x^\nu) x_\mu} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

invariant

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & - & - & - \\ -\gamma\beta_1 & - & - & \\ -\gamma\beta_2 & - & - & \\ -\gamma\beta_3 & - & - & - \end{pmatrix} \quad \text{with } \beta_i = \frac{v_i}{c}$$

Lorentz transformation matrix (not a tensor because doesn't transform like a tensor)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\boxed{\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu}$$

we can view entries of the $\Lambda^\mu{}_\nu$ matrix as derivatives of $(x')^\mu$ wrt $x^\nu$, i.e.

$$\frac{\partial x'^\mu}{\partial x^2} = \Lambda^\mu{}_2$$

## Vectors

$T^\mu(x)$ = contravariant vector – transforms like coordinates

$$\Rightarrow T^\mu(x) \xrightarrow[x \to x']{} T'^\mu(x') = \boxed{\frac{\partial (x')^\mu}{\partial x^\alpha}} T^\alpha(x)$$

Covariant vector – transforms like the gradient wrt $x^\mu$

$$☆ \quad \frac{\partial}{\partial x^\mu} = \partial_\mu = \left( \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \qquad \frac{\partial x^\alpha}{\partial (x')^\mu}$$

↳ transforms with the inverse of contravariant $(\Lambda^{-1})^\alpha{}_\mu$

scalar function: $f = f'(x')$

$$\boxed{\frac{\partial f(x)}{\partial x^\mu} \rightarrow \frac{\partial f(x')}{\partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial f(x)}{\partial x^\alpha}} \quad \frac{\partial x}{\partial x'}$$

now compare: $\frac{\partial}{\partial x^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial}{\partial x^\alpha}$

index up

$$\boxed{\partial^\alpha} = \frac{\partial}{\partial x_\alpha} = \eta^{\alpha\beta}\frac{\partial}{\partial x^\beta}$$

Contravariant

$$= \left(\frac{\partial}{\partial x^0}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}\right)$$

$$= \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

$$\boxed{\partial^\alpha = \frac{\partial}{\partial x_\alpha} \qquad \partial_\alpha = \frac{\partial}{\partial x^\alpha}}$$

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \sum_{\beta=0}^{3}\eta^{\alpha\beta}\frac{\partial}{\partial x^\beta} = \begin{cases} \alpha=0: & \frac{\partial}{\partial x^0} = \frac{\partial}{\partial x_0} \\ \alpha=1: & -\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x_1}\end{cases}$$

## Tensors

$T^{\mu\nu}$, $\mu = 0,\dots,3$, $\nu = 0,\dots,3$

Lorentz Tensor:

$$T^{\mu\nu}(x) \xrightarrow{x\to x'} T'^{\mu\nu}(x') = \sum_{\alpha,\beta=0}^{3} \underbrace{\left(\frac{\partial (x')^\mu}{\partial x^\alpha}\right)}_{\Lambda^\mu{}_\alpha}\underbrace{\left(\frac{\partial (x')^\nu}{\partial x^\beta}\right)}_{\Lambda^\nu{}_\beta} T^{\alpha\beta}(x)$$

↳ both indices transform like contravariant vectors (coordinates)

$\rightsquigarrow$ by $\frac{\partial (x')^\mu}{\partial x^\alpha} = \Lambda^\mu{}_\alpha$

From 2 vectors we can get a Lorentz invariant quantity.

$$\sum_{\alpha=0}^{3} A^\alpha B_\alpha \longrightarrow A'^\alpha(x')\, B'_\alpha(x') = \frac{\partial x'^\alpha}{\partial x^\mu} A^\mu(x')\frac{\partial x^\nu}{\partial (x')^\alpha} B_\nu(x)$$

$$= \underbrace{\frac{\partial (x')^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial (x')^\alpha}}_{\delta_\mu^\nu} A^\mu(x)\, B_\nu(x) = A^\mu(x) B_\mu(x)$$

$\Lambda^\alpha{}_\mu$ , $= \Lambda^{-1\nu}{}_\alpha$

column?

$$\sum_{\alpha=0}^{3}\Lambda^\alpha{}_\mu (\Lambda^{-1})^\nu{}_\alpha = (\Lambda\Lambda^{-1})\Big|_\mu^\nu = \mathbb{1}\Big|_\mu^\nu = \delta_\mu^\nu$$

row?

$\delta_\mu^\nu = \begin{cases} 1 & \text{if } \mu=\nu \\ 0 & \text{if } \mu\neq\nu\end{cases}$

$$\begin{array}{cc} A & B\downarrow \\ \to\begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} & \begin{pmatrix}5 & 6\\ 7 & 0\end{pmatrix}\end{array}$$

$$\sum_{\alpha=1}^{2} A^\alpha{}_1 B_\alpha{}^2 = 1\cdot 6 + 3\cdot 0 = (AB)_1^2$$

**d'Alembert operator** (wave eq.) $\rightsquigarrow$ Lorentz invariant because no indices left over after summation

$$\square = \sum_\alpha \partial^\alpha\partial_\alpha = \sum_{\alpha,\beta}\eta^{\alpha\beta}\partial_\alpha\partial_\beta = \partial_0^2 - \sum_i \partial_i^2 = \partial_t^2 - \nabla^2$$

# Free particle $(U=0)$

**Newtonian:** $L = \frac{1}{2}m\dot{\vec{x}}^2 - U = \frac{1}{2}m\dot{\vec{x}}^2$

action $S = \int \frac{1}{2}m\dot{\vec{x}}^2\,dt$

**Minkowski (SR):** time is no longer invariant (will be transformed) $\rightsquigarrow$ need new quantity

"proper time" $\tau$ = time measured by a comoving observer

$$ds^2 = c^2dt^2 - \underbrace{d\vec{x}^2}_{=0}$$

$= 0$ is the rest frame (for comoving observers)

$\boxed{d\tau^2 = dt^2_{rest}}$

invariant because $\boxed{d\tau = \frac{ds}{c}}$

$\hookrightarrow$ we want to define action using invariant quantities because we want laws of physics to be the same in all reference frames

$$c\,d\tau = ds = \sqrt{dt^2c^2 - d\vec{x}^2} = c\,dt\sqrt{1 - \frac{d\vec{x}^2}{dt^2c^2}} =$$

$$= c\,dt\sqrt{1 - \frac{v^2}{c^2}}$$

Consider $\boxed{S = -k\int c\,d\tau = -k\int c\sqrt{1-\frac{v^2}{c^2}}\,dt}$

Non-relativistic limit

$v \ll c$

$$-kc\sqrt{1-\frac{v^2}{c^2}} \underset{\substack{? \\ \text{around } \frac{v}{c}=0}}{\cong} \underbrace{(-kc)}_{\text{integration const.}} + k\frac{v^2}{2c} \Rightarrow mc\frac{v^2}{2} = \frac{1}{2}mv^2$$

$\boxed{k = mc}$

$$\Rightarrow \boxed{S = -mc^2\int\sqrt{1-\frac{v^2}{c^2}}\,dt} = \int L\,dt$$

$$\Rightarrow \boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}}} \boxed{= -\frac{mc^2}{\gamma}}.$$

# Relativistic Kinematics – 4velocity, 4momentum

$$\vec{p} \equiv \frac{\partial L}{\partial \dot{\vec{x}}} = -mc^2 \cdot \frac{\left(-\frac{2v^2}{c^2}\right)}{2\sqrt{1-\frac{v^2}{c^2}}} = m\dot{\vec{x}}\gamma = \gamma m\vec{v} \approx m\vec{v}$$

$v = \dot{x}$ ; $v <<< c$

$$H \equiv \dot{\vec{x}}\frac{\partial L}{\partial \dot{\vec{x}}} - L = \dot{\vec{x}}\gamma m\dot{\vec{x}} - \left(-\frac{mc^2}{\gamma}\right) = m\left(\gamma\dot{\vec{x}}^2 + \frac{c^2}{\gamma}\right) =$$

$$= \gamma m v^2 + \frac{mc^2}{\gamma} = m\gamma c^2 = E$$

non-relativistic limit: $\boxed{H = E \approx \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{mv^2}{2}}_{KE} + \dots}$

⇝ **4momentum:** $\boxed{p^\mu = \left(\frac{E}{c}, \vec{p}\right)}$

4vector, contravariant